# From curves to surfaces: a walk through Algebraic Geometry codes

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**CNRS; Université de Bordeaux**

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Some slides were taken from or largely inspired by J. Nardi's talk at the conference COGNAC. The speaker is very grateful to her.

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## Error–correcting codes

**GOAL:** retrieve the data which are lost/corrupted during transmission or storage.

Linear codes: $\mathbb{F}_q$–vector subspaces of $\mathbb{F}_q^n$ endowed with a metric. $d(\mathbf{x},\mathbf{0})=w(\mathbf{x})=\#\{x_i\neq 0\}$

$[n,k,d]_q$–code: code of length **n**, dimension **k** and minimum distance **d**. $d=\min_{\mathbf{x}\neq\mathbf{0}}\{w(\mathbf{x})\}$

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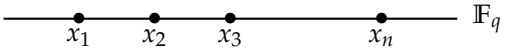


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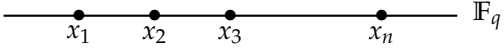


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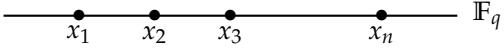


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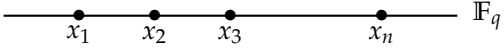


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⚠ **Drawback:** $n \leqslant q$.

The bigger the $q$, the less efficient the arithmetic.

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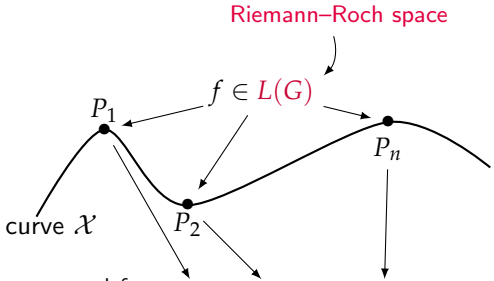


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*A divisor is a formal sum of points on $\mathcal{X}$*
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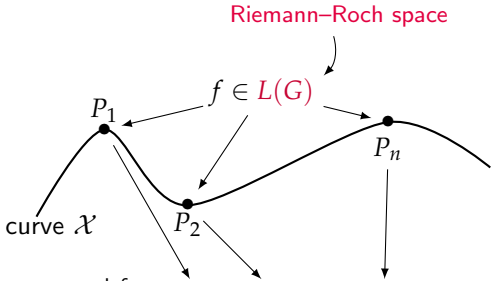


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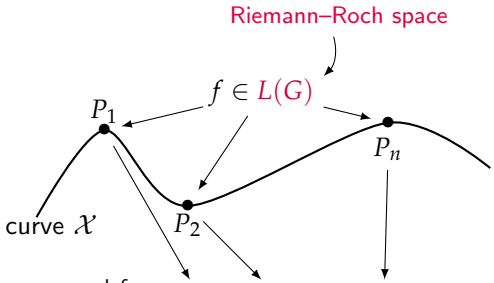


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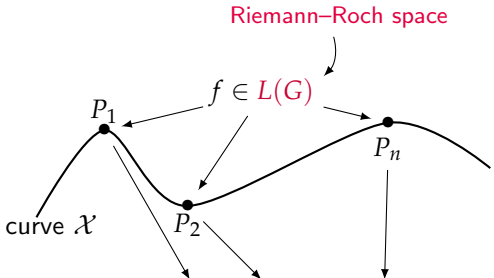


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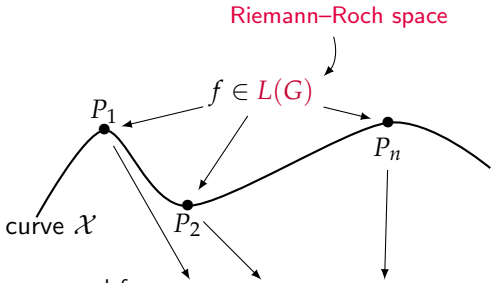


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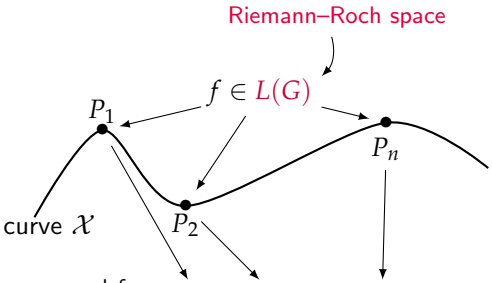


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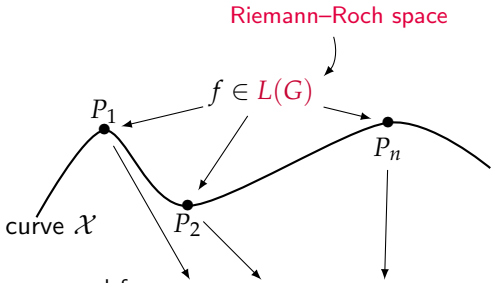


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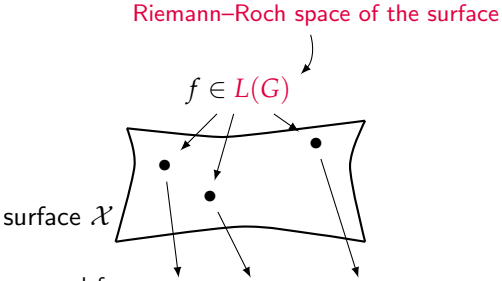


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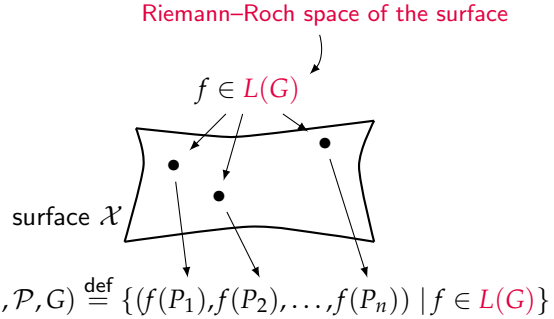


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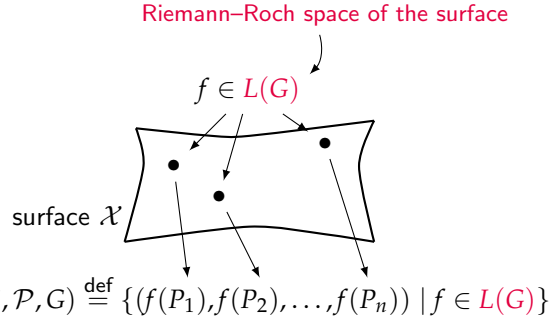


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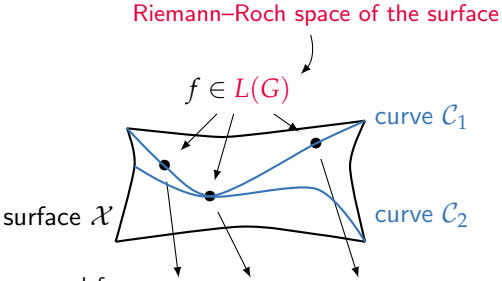


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$$\text{zeros}(f) \le \sum_{i=1}^{s_f} \#\mathcal{C}_i(\mathbb{F}_q)$$

To bound the minimum distance we need to

- bound $s_f$ (see VZ19, ABHP21a, ABHP21b)
- bound $\#\mathcal{C}(\mathbb{F}_q)$ for $\mathcal{C} \subset \mathcal{X}$ (see BN22)

## Have you ever seen an AG codes from higher–dimensional varieties? Maybe yes!

### Definition (Reed–Muller code)

*Let $N \geq 1$ and $r \geq 0$. We define the Reed–Muller code of order $r$ by*

$$\mathrm{RM}(N,r) = \{(f(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{F}_q^N} \mid f \in \mathbb{F}_q[X_1,\ldots,X_N]_{\leq r}\}.$$

For $r \leq q$, $\dim \mathrm{RM}(N,r) = \dim \mathbb{F}_q[X_1,\ldots,X_N]_{\leq r}$ and the minimum distance $d = q^N - rq^{N-1}$ is reached by product of linear factors.

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**Why is it an AG code?** ($N = 2 \rightsquigarrow$ AG code from a surface!)
Consider $\mathcal{X} = \mathbb{P}^N$ and $\mathcal{P} = \{(1,x_1,\ldots,x_N) \in \mathbb{P}^N(\mathbb{F}_q) \mid x_i \in \mathbb{F}_q\} = \mathbb{A}^N(\mathbb{F}_q) \simeq (\mathbb{F}_q)^N$.
Let $H$ be the hyperplane of $\mathbb{P}^N$ defined by $X_0 = 0$. Then, for any integer $r \geq 0$

$$L(rH) = \frac{1}{X_0^r} \cdot \mathbb{F}_q[X_0,\ldots,X_N]^{\mathsf{hom}}_{=r}.$$

Then $\mathrm{RM}(N,r) = C(\mathbb{P}^N,\mathcal{P},rH)$.

## (Non–exhaustive) Bibliography about AG codes from surfaces

- 1954: Reed–Muller codes
- 1986: Projective Reed–Muller (Lachaud) Parameters studied by Sorensen (1991)
- 1991: Restriction of RM Codes to projective algebraic varieties (Aubry)
- 1992: Quadric surfaces (Aubry)
- 2001: General study by Hansen
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- 2002: Toric varieties (Hansen) Higher–dimensional varieties
- 2005: Hermitian surface (Edoukou) Surfaces
- 2007: Exploring surfaces with small Picard rank (Zarzar)
- 2018: $\mathrm{rk}\mathrm{Pic}\mathcal{X} = 1$ or sectional genus $= 0$ (Little, Schenck)
- 2019: Hirzebruch surfaces (Nardi)
- 2020: Del Pezzo surfaces with Picard rank one (Blache et al.)
- 2021: Abelian surfaces (Aubry, **B.**, Herbaut, Perret)
- 2021: Surfaces without small genus curves and fibrations (Aubry, **B.**, Herbaut, Perret)

## Why AG codes? Typical asymptotical behavior of linear codes

For an $[n,k,d]$–code $C$, we define its *rate* $R \stackrel{\text{def}}{=} \frac{k}{n}$ and its *relative distance* $\delta \stackrel{\text{def}}{=} \frac{d}{n}$.

**Good** code: $R$ and $\delta$ close to 1.

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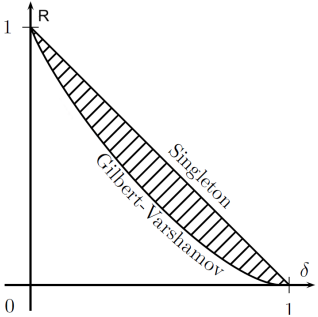

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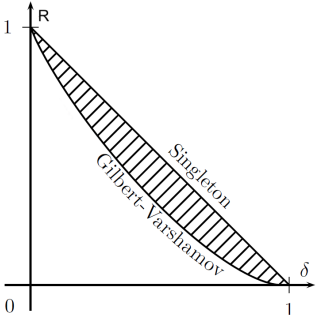


A random (linear) code of length $n$ and dimension $k$ satisfies $\frac{k}{n} \simeq 1 - H_q(\delta)$ with probability going to 1 when $n \to \infty$.

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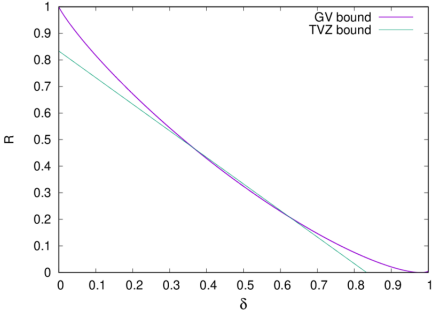

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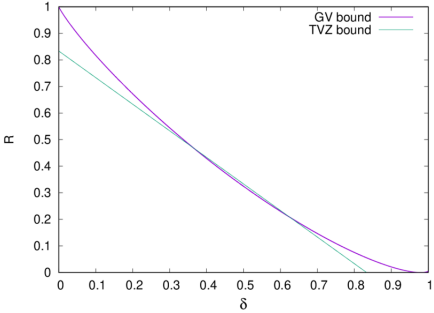


**AG codes from surfaces?**

✔ Working with **towers of surfaces** we could get longer codes

?? **Several invariants** come into play, things get more difficult!

A fist attempt has been done by Couvreur-Lebacque-Perret CLP21. **The game is still open...**

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   ✓ natural local decoding using curves on the surface

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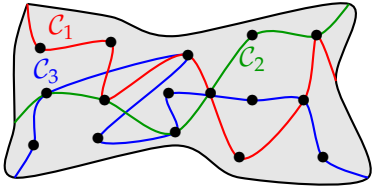

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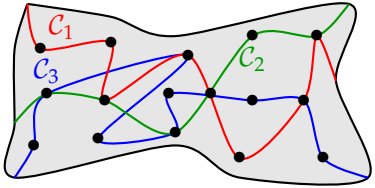


Then we have a procedure to decode a word $\boldsymbol{w}$ with respect to $C$.

1. Pick a curve $\mathcal{C}_i$ at random;
2. Use a decoding algorithm to decode $\boldsymbol{w}_{|\mathcal{C}_i}$ w.r.t. $C_{|\mathcal{C}_i}$ and replace the coordinates in $\boldsymbol{w}$;
3. Repeat 1 and 2 as many times as necessary so that for each $i$, $\boldsymbol{w}_{|\mathcal{C}_i} \in C_{|\mathcal{C}_i}$ ($\Rightarrow \boldsymbol{w} \in C$).

✔ Successfully applied to AG codes from cubic surfaces of $\mathbb{P}^3$ VZ10;
✖ May fail if too many errors gather on one curve;
✖ Characterizing codes from restrictions may be really hard.

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### Definition

*A code $C$ is said to be locally recoverable (LR) with locality $\ell$ if, for each $i \in \{1, \dots, n\}$, there is a subset $J_i \subseteq \{1, \dots, n\} \setminus \{i\}$, $\#J_i = \ell$ (called the recovery set), such that for any $c \in C$, we can recover the coordinate $c_i$ knowing the values $c_j$ for $j \in J_i$.*

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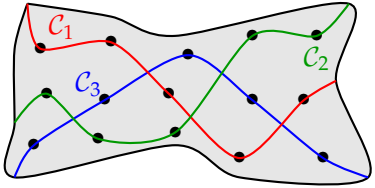


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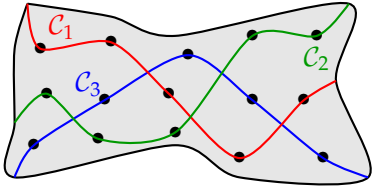


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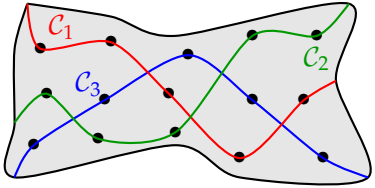


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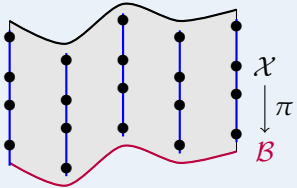
### LRCs on ruled surfaces — Salgado, Varilly-Alvarado, Voloch SVAV21




Fibers $\pi^{-1}(\{P\}) \simeq \mathbb{P}^1$ for every $P \in \mathcal{B}$.
Take $\mathcal{C}_i = \{$fibers of $\mathbb{F}_q$–points of $\mathcal{B}$ covering $\mathcal{P}\}$.

$\rightarrow$ Design codes from $\mathcal{X}$ whose restrictions to the $\mathcal{C}_i$ are Reed–Solomon codes of given degree.

## Quantum codes from classical linear codes: CSS codes

### Definition

*A quantum error-correcting code $Q$ is a subspace of $(\mathbb{C}^q)^{\otimes n}$. The length of $Q$ is $n$ and its dimension is the dimension as a subspace. We say that $Q$ has minimum distance $d$ if it can correct all quantum errors of weight less than or equal to $\lfloor (d-1)/2 \rfloor$.*

### Theorem (Calderbank and Shor '96, Steane '96)

Let $C_2 \subseteq C_1 \subseteq \mathbb{F}_q^n$ be linear codes of dimension $k_1$ and $k_2$. Let $d_1$ and $d_2^\perp$ denote the minimum distance of $C_1$ and $C_2^\perp$, respectively. Then, there exists a quantum code $Q_{C_1,C_2}$, called CSS code, with dimension $k_1 - k_2$ and minimum distance $d \geq \min\{d_1, d_2^\perp\}$.

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### Remark

Let $C$ be a self-orthogonal $[n,k,d]_q$ linear code i.e. $C \subseteq C^\perp$. Then taking $C_1 = C^\perp$ and $C_2 = C$ gives a CSS code of dimension $n - 2k$ and minimum distance $d^\perp$.

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**Quantum codes from surfaces:**

✔ Constructions have been done in particular cases (toric surfaces Han12)

✖ Dealing with the dual code is much more difficult in general

the orthogonal of an AG code from a surface is not an AG code from the surface

In a series of papers Couvreur studied the duality theory of AG codes from surfaces.

**The game is still open...**

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- ✔ provide long codes with nice properties
- ✔ can be effectively implemented
- ✔ have many applications (quantum codes, LRCs, McEliece cryptosystem, Proof of Knowledge...)
  (Leen Demuys's talk!) (Ruben De Smet's talk!)

### AG codes from surfaces

- ✔ allow to construct longer codes
- ✔ have a geometric structure richer than curves, which give interesting properties (e.g. locality),
- ✖ lack of generic algorithms to encode and decode

**We should study AG codes from surfaces and**

- rise to the challenge of implementing them;
- exploring families of surfaces giving good (classical and quantum) codes.

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**We should study AG codes from surfaces and**

- rise to the challenge of implementing them;
- exploring families of surfaces giving good (classical and quantum) codes.


# Thank you for your attention!

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