Algebraic Geometry codes in the sum-rank metric

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https://arxiv.org/abs/2303.08903

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 899987

Linear codes and codes in the Hamming metric

k a field (think about $k = \mathbb{F}_q$), $\mathcal{H}$ a k-linear vector space endowed with a metric

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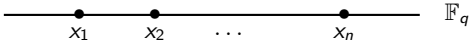

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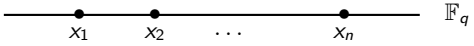

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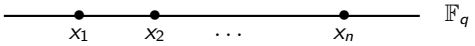

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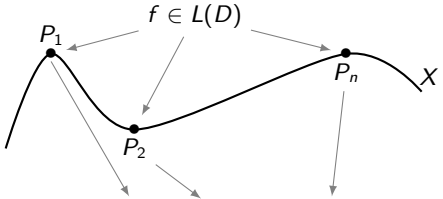

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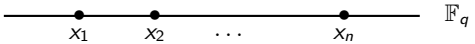

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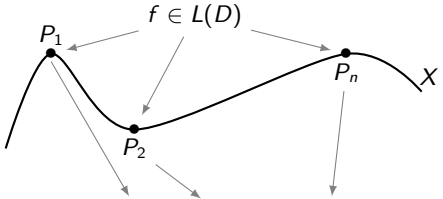

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- ✔ **Good parameters:** $n+1-g \leq \delta+d \leq n+1$
- ✔ **Longer codes**

General definitions

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$n_i = 1 \ \forall i \quad \rightsquigarrow \quad$ codes of length s in the Hamming metric

Particular case and Singleton bound

$\ell =$ finite extension of k of degree r

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Singleton bound

The ℓ-parameters of $\mathcal{C}$ satisfy

$$d + \delta_\ell \leq n_\ell + 1.$$

Codes with parameters attaining this bound are called Maximum Sum-Rank Distance (MSRD).

Ore polynomials and Linearized Reed–Solomon codes

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As in the Hamming case, we can **try to overcome the problem using algebraic curves**
Main idea: consider Ore polynomials with coefficients in the function field of a curve

Divisors and Riemann–Roch spaces: classical theory

Definition

Let X be a nice curve, K its function field. A divisor on X is a formal finite sum

$$D = \sum_{\mathfrak{p} \in X} n_{\mathfrak{p}} \mathfrak{p} \quad \text{with } n_{\mathfrak{p}} \in \mathbb{Z} \text{ almost all zero.}$$

The group of divisors on X is denoted by $Div(X)$.
$D \in Div(X)$ is positive, $D \geq 0$, if $n_{\mathfrak{p}} \geq 0 \ \forall \mathfrak{p}$. The degree of D is $\deg_X(D) = \sum_{\mathfrak{p} \in X} n_{\mathfrak{p}} \deg_X(\mathfrak{p})$.

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Riemann–Roch theorem

Let K_X denotes a canonical divisor on X. For any divisor $D \in \mathrm{Div}(X)$ we have

$$\dim_k L_X(D) = \deg_X(D) + 1 - g_X + \underbrace{\dim_k L_X(K_X - D)}_{=0 \text{ when } \deg_X(D) > 2g_X - 2.},$$

Our setting

$$\begin{tikzcd} Y \arrow[d, "\pi"] \\ X \end{tikzcd}$$

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For $x \in K^\times$, consider the algebra

$$D_{L,x} := L[T;\Phi]/(T^r - x)$$

and for all $\mathfrak{p} \in X$, the algebras $D_{L_\mathfrak{p},x} := K_\mathfrak{p} \otimes_K D_{L,x} = L_\mathfrak{p}[T;\Phi]/(T^r - x)$.

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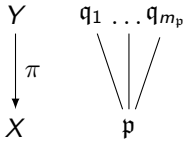

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$L := k(Y)$ the fields of functions of Y, $\mathrm{Gal}(L/K) = \langle \Phi \rangle$

For $\mathfrak{p} \in X$ we have the decomposition $L_\mathfrak{p} := K_\mathfrak{p} \otimes_K L \simeq \prod_{\mathfrak{q}|\mathfrak{p}} L_\mathfrak{q}$.

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- ☐ principal divisors associated to $f \in D_{L,x}$
  $\rightsquigarrow$ need to define a valuation
- ☐ Riemann–Roch spaces of $D_{L,x}$
- ☐ a Riemann–Roch theorem
- ☐ equivalent of "evaluate at a rational point"

Our setting

$$\begin{tikzcd} Y \arrow[d, "\pi"] & \mathfrak{q}_1 \dots \mathfrak{q}_{m_\mathfrak{p}} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \\ X & \mathfrak{p} \end{tikzcd}$$

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Define the valuation map $w_{\mathfrak{q}_j,x} : D_{L_\mathfrak{p},x} \to \frac{1}{r}\mathbb{Z} \sqcup \{\infty\}_{(1 \le j \le m_\mathfrak{p})}$: for $f = f_0 + f_1 T + \dots + f_{r-1} T^{r-1}$,

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Divisors and Riemann–Roch spaces over Ore polynomial rings

Definition (Riemann–Roch spaces of $D_{L,x}$)

Let $E = \sum_{\mathfrak{q} \in Y} n_{\mathfrak{q}} \mathfrak{q} \in Div_{\mathbb{Q}}(Y) := Div\,(Y) \otimes \mathbb{Q}$, *with* $n_{\mathfrak{q}} \in \frac{1}{b_{\mathfrak{p}}}\mathbb{Z}$ *where* $\mathfrak{p} = \pi(\mathfrak{q})$.
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Riemann's inequality for $\Lambda_{L,x}(E)$

For a divisor $E = \sum_{\mathfrak{q}\in Y} n_{\mathfrak{q}}\mathfrak{q} \in \mathrm{Div}_{\mathbb{Q}}(Y)$ the space $\Lambda_{L,x}(E)$ is finite dimensional over k and

$$\dim_k \Lambda_{L,x}(E) \geq r \cdot \deg_Y(E) - r \cdot (g_Y - 1) - \frac{r^2}{2} \sum_{\mathfrak{p}\in X} \frac{b_{\mathfrak{p}}-1}{b_{\mathfrak{p}} e_{\mathfrak{p}}} \deg_X(\mathfrak{p}).$$

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Linearized Algebraic Geometry codes (B., Caruso, 2023)

Let $E = \sum_{\mathfrak{q} \in Y} n_\mathfrak{q}\mathfrak{q} \in \mathrm{Div}_\mathbb{Q}(Y)$. Chose $x \in K$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ rational places on X such that the hypotheses hold. Consider

$$\begin{array}{rccc} \alpha: & \Lambda_{L,x}(E) & \longrightarrow & \prod_{i=1}^s \mathrm{End}_k(V_{\mathfrak{p}_i}) \\ & f & \mapsto & \left(\overline{\varepsilon}_{\mathfrak{p}_i}(f)\right)_{1 \le i \le s}. \end{array}$$

The code $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ is defined as the image of α.

Code's parameters

We study the parameters of the k-linear code $\mathcal{C}$ in $\prod_{i=1}^{s} \mathrm{End}_k(V_{\mathfrak{p}_i})$.

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Theorem (B., Caruso, 2023)

Assume $\deg_Y(E) < sr$. Assume the previous hypotheses and that $D_{L,x}$ contains no nonzero divisors. Then, the dimension δ and the minimum distance d of $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ satisfy

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Singleton bound: $rd + \delta \leq n + r$

We have: $rd + \delta \geq n + r - \left(r \cdot g_Y + \frac{r^2}{2} \sum_{\mathfrak{p} \in X} \frac{b_{\mathfrak{p}} - 1}{b_{\mathfrak{p}} e_{\mathfrak{p}}} \deg_X(\mathfrak{p}) \right)$

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$X = \mathbb{P}^1_k$, $Y = \mathbb{P}^1_\ell$, $E = \frac{\delta}{r} \cdot \infty \in \operatorname{Div}_\mathbb{Q}(Y)$ $\rightsquigarrow$ linearized Reed–Solomon codes!
Our lower bounds $\Rightarrow$ MSRD codes

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Thanks for your attention!

Questions?

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