Algebraic Geometry codes in the sum-rank metric

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Table of Contents

Linear codes and codes in the Hamming metric

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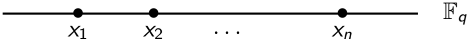

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(Singleton bound: $\delta + d \leq n + 1$)

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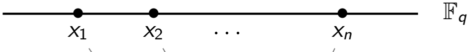

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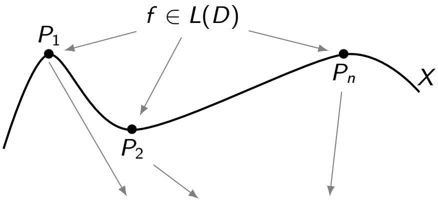

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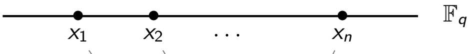

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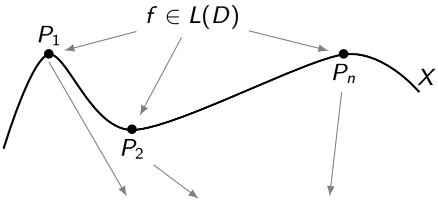

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- ✔ **Good parameters:** $n+1-g \leq d+\delta \leq n+1$
- ✔ **Longer codes**

General definitions

$\underline{V} = (V_1, \ldots, V_s)$ s-uple of k-vector spaces $(n_i = \dim_k V_i)$

$$\underbrace{\mathcal{H} = \mathrm{End}_k(\underline{V})}_{\text{k-vector space of dimension } \sum_{i=1}^{s} n_i^2} \quad := \quad \underbrace{\mathrm{End}_k(V_1) \times \cdots \times \mathrm{End}_k(V_s)}_{\text{k-linear morphisms } V_i \to V_i}$$

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Definition

Let $\underline{\varphi} = (\varphi_1, \dots, \varphi_s) \in \mathcal{H}$*. The sum-rank weight of* $\underline{\varphi}$ *is* $w_{srk}(\underline{\varphi}) := \sum_{i=1}^{s} rk(\varphi_i)$*.*
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$n_i = 1 \ \forall i \quad \rightsquigarrow \quad$ codes of length s in the Hamming metric

Particular case and Singleton bound

ℓ = finite extension of k of degree r

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$\rightsquigarrow$ ℓ–variants of the parameters:

$$\begin{cases} n_\ell := rs & \ell\text{–length} \\ \delta_\ell := \dim_\ell \mathcal{C} & \ell\text{–dimension} \\ \text{the minimum distance stays unchanged} \end{cases}$$

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Singleton bound

The ℓ-parameters of $\mathcal{C}$ satisfy
$$d + \delta_\ell \leq n_\ell + 1.$$

Codes with parameters attaining this bound are called Maximum Sum-Rank Distance (MSRD).

Ore polynomials and Linearized Reed–Solomon codes (Martínez-Peñas, 2018)

ℓ field, $\Phi : \ell \to \ell$ ring homomorphism, $\ell^{\Phi=1} = k$, $[\ell : k] = r$

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The ring of Ore polynomials $\ell[T;\Phi]$ is the ring whose elements are polynomials with coefficients in ℓ, with usual $+$ and

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length $=$ rs dimension $= \delta$ minimum distance $= rs - \delta + 1$ $\Rightarrow$ MSRD codes

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Main idea

Consider Ore polynomials with coefficients in the function field of a curve

Notations

Consider a smooth projective irreducible algebraic curve X of genus g_X defined over k

$K = k(X)$ - function field of X

$X^\star$ - set of places (or, equivalently, closed points) of X

for $\mathfrak{p} \in X^\star$, set

$\mathcal{O}_\mathfrak{p}$ - the ring of integers of $\mathfrak{p}$

$k_\mathfrak{p}$ - the residue class field of $\mathfrak{p}$

$\deg_X(\mathfrak{p})$ - the degree of $\mathfrak{p}$, the degree of the extension $k_\mathfrak{p}/k$

$K_\mathfrak{p}$ - the completion of K at $\mathfrak{p}$, equipped with the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$

Divisors and Riemann–Roch spaces: classical theory

Definition

A divisor on X is a formal finite sum

$$D = \sum_{\mathfrak{p} \in X^\star} n_\mathfrak{p} \mathfrak{p} \quad \textit{with } n_\mathfrak{p} \in \mathbb{Z} \textit{ almost all zero.}$$

The group of divisors on X is denoted by $Div(X)$.
$D \in Div(X)$ is positive, $D \geq 0$, if $n_\mathfrak{p} \geq 0\ \forall \mathfrak{p}$. The degree of D is $\deg_X(D) = \sum_{\mathfrak{p} \in X^\star} n_\mathfrak{p} \deg_X(\mathfrak{p})$.

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$$L_X(D) := \{x \in K^\times \mid (x) + D \geq 0\} \cup \{0\},$$

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Riemann–Roch theorem

Let K_X denotes a canonical divisor on X. For any divisor $D \in \mathrm{Div}(X)$ we have

$$\dim_k L_X(D) = \deg_X(D) + 1 - g_X + \underbrace{\dim_k L_X(K_X - D)}_{=0 \text{ when } \deg_X(D) > 2g_X - 2.},$$

Our setting

$$\begin{array}{c} Y \\ \Big\downarrow \pi \\ X \end{array}$$

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For $x \in K^\times$, consider the algebra

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Divisors and Riemann–Roch spaces over Ore polynomial rings

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Riemann's inequality for $\Lambda_{L,x}(E)$

For a divisor $E = \sum_{\mathfrak{q}\in Y^\star} n_\mathfrak{q}\mathfrak{q} \in \mathrm{Div}_{\mathbb{Q}}(Y)$ the space $\Lambda_{L,x}(E)$ is finite dimensional over k and

$$\dim_k \Lambda_{L,x}(E) \geq r\cdot\deg_Y(E) - r\cdot(g_Y - 1) - \frac{r^2}{2}\sum_{\mathfrak{p}\in X^\star} \frac{b_\mathfrak{p}-1}{b_\mathfrak{p} e_\mathfrak{p}} \deg_X(\mathfrak{p}).$$

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Definition (Linearized Algebraic Geometry codes)

Let $E = \sum_{\mathfrak{q} \in Y^\star} n_\mathfrak{q}\mathfrak{q} \in Div_\mathbb{Q}(Y)$. Chose $x \in K$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ rational places on X such that the hypotheses hold. Set $V_{\mathfrak{p}_i} := \mathcal{O}_{L_{\mathfrak{p}_i}}/t_{\mathfrak{p}_i}\mathcal{O}_{L_{\mathfrak{p}_i}}$. Consider

$$\begin{array}{rccc} \alpha : & \Lambda_{L,x}(E) & \longrightarrow & \prod_{i=1}^s End_k(V_{\mathfrak{p}_i}) \\ & f & \mapsto & \left(\overline{\varepsilon}_{\mathfrak{p}_i}(f)\right)_{1 \le i \le s}. \end{array}$$

The code $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ is defined as the image of α.

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Theorem (B., Caruso)

Assume $\deg_Y(E) < sr$. Assume the previous hypotheses and that $D_{L,x}$ contains no nonzero divisors. Then, the dimension δ and the minimum distance d of $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ satisfy

$$\delta \geq r\cdot \deg_Y(E) - r\cdot(g_Y - 1) - \frac{r^2}{2} \sum_{\mathfrak{p}\in X^\star} \frac{b_{\mathfrak{p}} - 1}{b_{\mathfrak{p}} e_{\mathfrak{p}}} \deg_X(\mathfrak{p}),$$

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Singleton bound: $rd + \delta \leq n + r$

We have: $rd + \delta \geq n + r - \left(r \cdot g_Y + \frac{r^2}{2} \sum_{\mathfrak{p} \in X^\star} \frac{b_\mathfrak{p} - 1}{b_\mathfrak{p} e_\mathfrak{p}} \deg_X(\mathfrak{p}) \right)$

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$$\deg_Y(E') \leq -\sum_{i=1}^{s} d_i + \sum_{\mathfrak{q} \in Y^\star} \frac{r \cdot n_{\mathfrak{q}}}{e_{\mathfrak{p}}\, m_{\mathfrak{p}}} \deg_X(\pi(\mathfrak{q})) = \omega - sr + \deg_Y(E).$$

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If $\omega < sr - \deg_Y(E) \Rightarrow \operatorname{Nrd}(f) = 0 \Rightarrow \mu_f$ is not injective $\Rightarrow$ f is a nonzero zero divisor in $D_{L,x}$

Sketch of the proof

Want: $d \geq sr - \deg_Y(E)$ + bound on the dimension δ

Let $0 \neq f \in \Lambda_{L,x}(E)$, with $\omega = w_{\mathrm{rk}}(\alpha(f)) = \sum_{i=1}^{s} \operatorname{rk} \bar{\varepsilon}_{\mathfrak{p}_i}(f)$.

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Injectivity of the map $\alpha \Rightarrow \delta = \dim_k \Lambda_{L,x}(E) \rightsquigarrow$ lower bound on δ via Riemann's inequality ✔

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ℓ-parameters of the code

For the code $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ with $x, \mathfrak{p}_1, \ldots, \mathfrak{p}_s, E$ satisfying the hypotheses, we have

- $n_\ell = sr$,
- $\delta_\ell \geq \deg_Y(E) - r\cdot(g_X - 1) - \frac{r}{2} \sum_{\mathfrak{p} \in X^\star} \frac{b_\mathfrak{p} - 1}{b_\mathfrak{p}} \deg_X(\mathfrak{p})$,
- $d \geq sr - \deg_Y(E)$.

Linearized AG codes over $\mathbb{P}^1$ are Linearized Reed–Solomon codes

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Choose the function $x = t \in K^\times = k(t)^\times$. Then

$$b_\mathfrak{p} = \begin{cases} r \text{ for } \mathfrak{p} = 0, \infty, \\ 1 \text{ for all other } \mathfrak{p} \in X^\star, \end{cases}$$

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Our lower bounds: $\delta_\ell \geq m + 1$ and $d \geq sr - m = n_\ell - m \Rightarrow$ MSRD codes

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Merci de votre attention !

Questions?

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