On the number of rational points of curves over a surface in $\mathbb{P}^3$

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Curves, surfaces, rational points and all that jazz

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An algebraic projective variety X defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ irreducible over $\mathbb{F}_q$:

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Today: algebraic varieties of dimension one (curves C) and two (surfaces S) in $\mathbb{P}^3$.

Existing bounds

Theorem [Hasse–Weil, 1948]

If C is an absolutely irreducible smooth curve of genus g defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$.

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Theorem [Hasse–Weil, 1948, Aubry–Perret, 1993]

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Theorem [Homma, 2012]

If C is a non–degenerate curve defined over $\mathbb{F}_q$ of degree δ in $\mathbb{P}^n$, with $n \geq 3$, then $\#C(\mathbb{F}_q) \leq (\delta - 1)q + 1$.

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Theorem [Stöhr–Voloch, 1986]

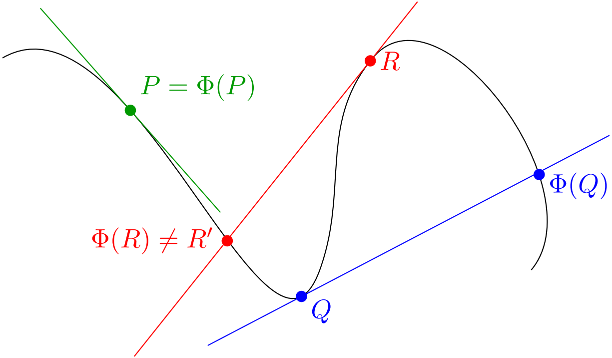
Let $C/\mathbb{F}_q$ be an irreducible smooth curve of genus g and degree δ in $\mathbb{P}^n$. Let $\nu_1, \ldots, \nu_{n-1}$ be its Frobenius orders (generically $\nu_i = i$). Then

$$\#C(\mathbb{F}_q) \leq \frac{1}{n}\left((\nu_1 + \cdots + \nu_{n-1})(2g-2) + (q+n)\delta\right).$$

Stöhr and Voloch's strategy for plane curves

Take C a plane curve of deg. δ defined by $f = 0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

$$\begin{array}{rcl} C(\mathbb{F}_q) & = & \{P \in C \mid \Phi(P) = P\} \\ & & \quad\quad\quad \cap\!\!| \\ & & \{P \in C \mid \Phi(P) \in T_P C\} \overset{\text{def}}{=} \mathcal{Z}. \end{array}$$



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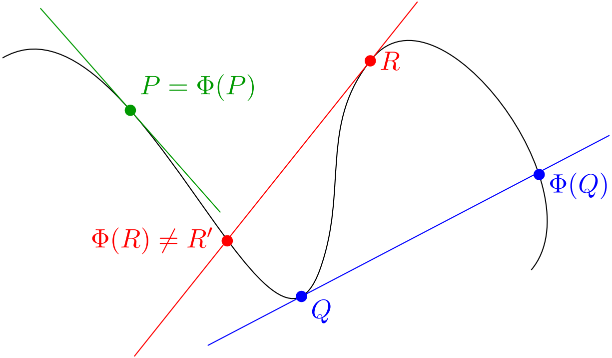

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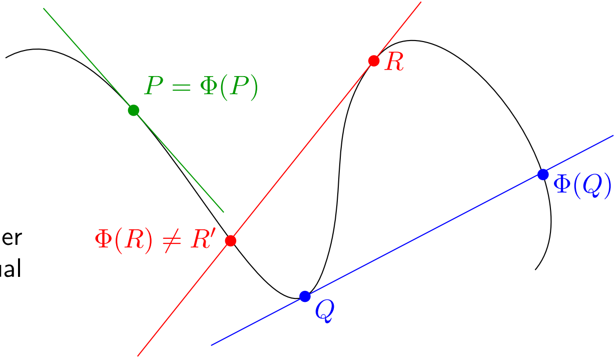

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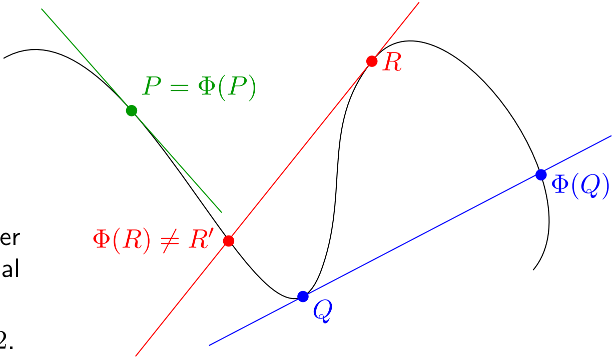

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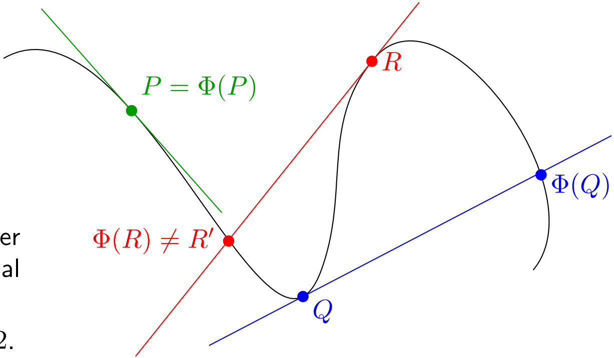

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Theorem [Stöhr–Voloch, 1986]

If C has at least a non–flex point $(\Rightarrow \dim \mathcal{Z} = 0)$, then $\#C(\mathbb{F}_q) \leq \frac{1}{2}\delta(\delta+q-1)$.

Ideas & Motivations

Let $C \subset S \hookrightarrow \mathbb{P}^n$ (via a very ample divisor).

Goal: bounding $\#C(\mathbb{F}_q)$ in terms of the embedding.

(features of the surface S and the ambient $\mathbb{P}^n$)

Main motivations:

- New bound for the number of rational points on projective curves.

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- Application to geometric coding theory.

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Bounding the minimum distance of a code from a surface S	$\rightsquigarrow$	Bounding $\#C(\mathbb{F}_q)$ for the irreducible curves C on S
Better lower bound for the minimum distance	$\Longleftrightarrow$	Better upper bound for $\#C(\mathbb{F}_q)$

Strategy ($n = 3$)

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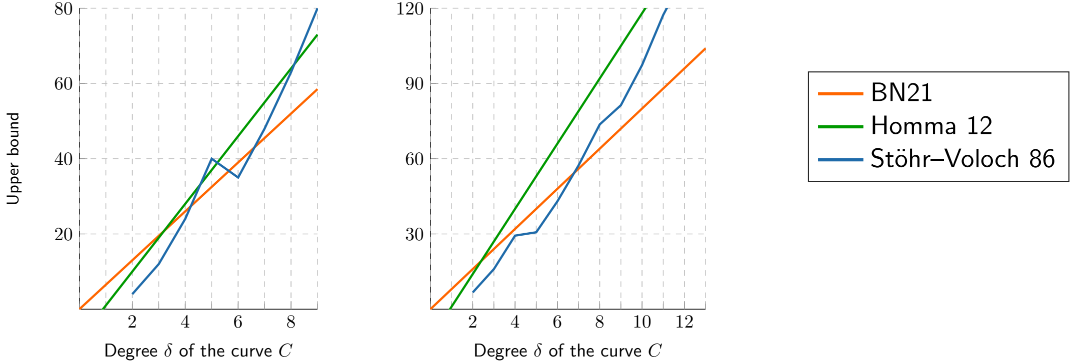
If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min\limits_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

Comparisons with pre–existing bounds



(a) $q = 9$ and $d = 5$ (b) $q = 13$ and $d = 4$

Figure: Bounds on the number of $\mathbb{F}_q$–points on a non–plane curve C on a degree d surface $S \subset \mathbb{P}^3$.

$\rightarrow$ It is worth working on this bound!

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Aim: understanding the components of the curve C_Φ^S for a Frobenius classical surface.

Osculating spaces and P-orders (Stöhr–Voloch theory 1)

Let $C \subset \mathbb{P}^3$ be an absolutely irreducible projective curve defined over $\mathbb{F}_q$. Fix $P \in C$.
An integer j is a P–order if there exists a plane intersecting the curve C with multiplicity j at P.
If C is non–plane and P is non–singular, there are exactly four distinct P–orders:

$$j_0 = 0 < j_1 < j_2 < j_3.$$

Remark: $j_1 = 1 \Leftrightarrow C$ is non–singular at the point P.

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Osculating spaces: $T_P^{(i)}C = \bigcap\{\text{planes } H \text{ s.t. } m_P(C,H) \geq j_{i+1}\}$.

Equation of the osculating plane $T_P^{(2)}C$: $\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$

where $D_t^{(j)}$ are the *Hasse derivatives* with respect to a a local parameter t at P defined by

$$D_t^{(i)} t^k = \binom{k}{i} t^{k-i}.$$

Frobenius orders (Stöhr–Voloch theory 2)

Fix $P \in C \subset \mathbb{P}^3$ with P–orders $(0, j_1, j_2, j_3)$. Then $\Phi(P) \in T_P^{(2)} C$ if and only if

$$\Delta(j_1, j_2) \overset{\text{def}}{=} \begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)} x_0 & D_t^{(j_1)} x_1 & D_t^{(j_1)} x_2 & D_t^{(j_1)} x_3 \\ D_t^{(j_2)} x_0 & D_t^{(j_2)} x_1 & D_t^{(j_2)} x_2 & D_t^{(j_2)} x_3 \end{vmatrix} = 0$$

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Theorem [Stöhr–Voloch, 1986]

There exist integers $\nu_1 < \nu_2$ s.t. $\Delta(\nu_1, \nu_2)$ is a nonzero function.

Definition

The integers $\nu_0 = 0, \nu_1, \nu_2$ chosen minimally with respect to the lexicographic order are called the Frobenius orders of C.

The curve C is Frobenius classical if $(\nu_1, \nu_2) = (1, 2)$, Frobenius non–classical otherwise.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

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Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

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Frobenius classical components of C_Φ^S

Recap: A component of C_Φ^S falls in one of the following cases:

- $\nu_1 > 1$: in this case, if it has $\delta \leq q$, it is plane;
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Example of surface with highly reducible C_Φ^S

Over $\mathbb{F}_5$, consider the surface S defined by

$$\begin{aligned} f = \quad & 2X_0X_1^2 + 2X_1^3 + 2X_0^2X_2 + 2X_0X_1X_2 + X_1^2X_2 + 2X_0X_2^2 + 3X_1X_2^2 \\ & +3X_2^3 + 4X_0^2X_3 + X_0X_1X_3 + X_1^2X_3 + 2X_1X_2X_3 + 2X_2^2X_3 \\ & +3X_0X_3^2 + 4X_1X_3^2 + X_2X_3^2. \end{aligned}$$

The curve C_Φ^S has degree 21 and is formed of 15 $\mathbb{F}_5$–lines and one non–plane sextic $(\delta = q + 1)$.

Main result & Conclusion

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- **Embedding entails arithmetic and geometric constraints on a variety:** For $\delta = 11$ and $d = 5$ over $\mathbb{F}_9$, C has genus at most 17 and $\#C(\mathbb{F}_q) \leq 72$. In `ManyPoints`, maximal curves of genus 16 and 17 have 74 $\mathbb{F}_9$–points. These record curves cannot lie on a Frobenius classical surface in $\mathbb{P}^3$, unless being a component of C^S_Φ.

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Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
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Γ_C and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
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Idea: Fix this dimension incompatibility by blowing up $\mathcal{T}_S$ or $S \times S$.