Weil polynomials of abelian varieties over finite fields with many rational points

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Maximality: under a certain order relation (defined later).

Spoiler

Theorem (B., Giangreco '22)

Let g be an positive integer and let $r_1, \dots, r_g$ be the roots of the polynomial $\mathfrak{f}_g$. Then, there exists a real number c_g such that for any $q = p^{2e} > c_g$ (q is an even power of a prime p), coprime with $\mathfrak{f}_g(0)$, the isogeny class $\mathcal{I}^0_{max}(g, q)$ exists and has Weil polynomial $h_g(t, \sqrt{q})$, where

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Corollary (B., Giangreco '22)

$\mathcal{I}^0_{max}(g,q)$ is ℓ-cyclic for all prime numbers ℓ that do not divide

$$N_g := \mathfrak{f}_g(4)\mathfrak{f}_g(0)\Delta_g,$$

where Δ_g is the discriminant of $\mathfrak{f}_g$.

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Abelian varieties and their groups of rational points intervene in cryptography and geometric coding theory

Abelian varieties: first step

Definition

An abelian variety A defined over a field k is a connected and completed variety with a group structure over $A(\bar{k})$. It is called simple if it does not contain proper abelian sub-varieties $\neq 0$.

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An isogeny is a surjective homomorphism from A to B with $\dim(A) = \dim(B)$. An isogeny from A to B implies the existence of an isogeny from B to A. This defines an equivalence relation. We say that A and B are isogenous, $A \sim B$. We denote $\mathcal{A}$ an isogeny class.

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Theorem (Poincaré Splitting Theorem)

An abelian variety A defined over a field k is (uniquely) isogenous to the product

$$A \sim B_1^{e_1} \times \cdots \times B_n^{e_n}$$

where the abelian varieties B_i are simple and pairwise non isogenous over k.

Abelian varieties: second step

$f_A(t)$: the characteristic polynomial of the Frobenius endomorphism

Multiplication map:

$$m_A : A \to A$$
$$P \mapsto mP$$

p-rank and ordinary varieties:
Let $q = p^r$, we define

$$A[p](\overline{\mathbb{F}}_q) := \ker(p_A).$$

We call the p-rank of A the dimension of $A[p](\overline{\mathbb{F}}_q)$.

Definition

An abelian variety with maximal p-rank is called ordinary.

Weil polynomials

Definition

Let $q = p^r$. A Weil q-polynomial is a monic even degree polynomial with integer coefficients, whose all roots are algebraic integers of absolute value $\sqrt{q}$. Over the real numbers, it is of the form

$$\prod_i (t^2 + x_i t + q), \quad x_i \in \mathbb{R} \text{ and } |x_i| \leq 2\sqrt{q}.$$

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Tate: $A \sim B \iff f_A(t) = f_B(t)$

Hence, we can talk about the Weil polynomial of an isogeny class $\mathcal{A}$:

$$f_{\mathcal{A}}(t)$$

Classical and ordinary Honda–Tate theory

Honda–Tate theory

$$\left\{\begin{array}{c}\textit{isogeny classes of simple abelian}\\ \textit{varieties defined over } \mathbb{F}_q\end{array}\right\} \iff \{\textit{irreducible Weil } q\textit{-polynomials}\}$$

⚠ not all irreducible Weil q-polynomials are Weil polynomials of a simple isogeny class;
⚠ not all simple isogeny classes have an irreducible Weil polynomial.

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✔ The Weil polynomial of a simple isogeny class is

$$f_{\mathcal{A}}(t) = h(t)^e,$$

for $h(t) \in \mathbb{Z}[t]$ an irreducible q-polynomial and e a positive integer.

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Honda–Tate theory (Ordinary)

An ordinary and irreducible Weil q-polynomial is always the Weil polynomial of a simple ordinary isogeny class.

What information do we get from the Weil polynomial?

The cardinality of the group of rational points:

$$\#A(\mathbb{F}_q) = f_A(1)$$

The property of being ordinary:

$$\mathcal{A} \text{ is ordinary} \iff f_{\mathcal{A}}(t) \text{ is an ordinary Weil } q\text{-polynomial}$$

The cyclicity of the group of rational points[1]: for a prime number ℓ we have

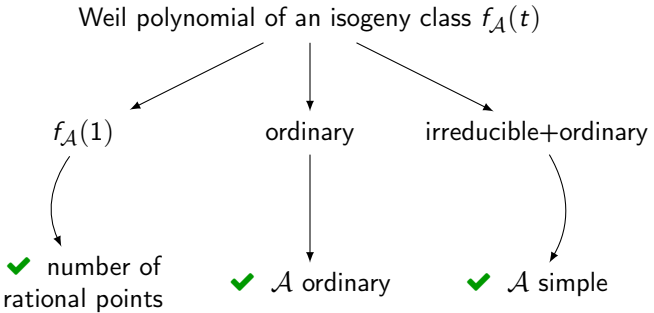
$$\mathcal{A} \text{ is } \ell\text{-cyclic} \iff \ell \text{ does not divide } (\widehat{f_{\mathcal{A}}(1)}, f'_{\mathcal{A}}(1)),$$

where:

- an isogeny class $\mathcal{A}$ is called ℓ-cyclic if $A(\mathbb{F}_q)_\ell$ is cyclic for any $A \in \mathcal{A}$, $A(\mathbb{F}_q)_\ell$ being the ℓ-part of the group of rational points of A;
- for an integer z, $\widehat{z}$ denotes the quotient of z by its radical.

[1] A. Giangreco–Maidana, Finite Fields Appl., 57 (2019).

So far, so good?



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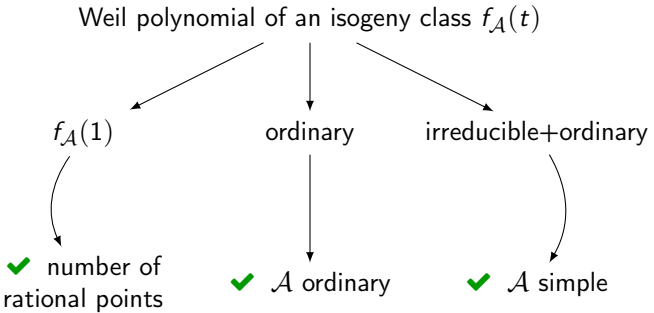

✔ cyclicity criterion for isogeny classes

Algebraic integers enter the game

Definition

An algebraic integer is a complex number that is the root of a monic polynomial with coefficients in $\mathbb{Z}$. It is called totally positive if all its conjugates are positive real numbers.

Ann. Inst. Fourier, Grenoble
33, 3 (1984), 1-28

TOTALLY POSITIVE ALGEBRAIC INTEGERS OF SMALL TRACE

by Christopher SMYTH

Let $r \geqslant 0$ be a given integer. We describe an algorithm for finding all totally positive algebraic integers α which satisfy

$$\mathrm{Tr}\,\alpha - \deg\alpha = r \qquad (1)$$

(where $\mathrm{Tr}\,\alpha$ = trace of α, $\deg\alpha$ = degree of α). That r must be non-negative is an immediate consequence of the inequality of the arithmetic and geometric means. The algorithm is based on a recent improvement [5] of a result of Siegel [3], combined with a method of Robinson [1] for enumerating totally real polynomials of a specific type. The algorithm was implemented on the University College, Cardiff, Honeywell computer which took 40 minutes CPU time to find all relevant α with $r = 0, 1, 2, \ldots, 6$. (Almost all of this time was spent on the last case: $r = 6$, $\deg\alpha = 7$). The table of these α appears as an appendix to this paper.

This work was stimulated by a question of Serre, who asked for a list of these algebraic integers, for an application connected with bounding the number of points on algebraic curves over finite fields.

Minimal polynomials of algebraic integers

$\mathcal{F}_g$: the set of minimal polynomial of totally positive algebraic integers of degree g;

that is

the set of monic polynomials of degree g with integer coefficients, irreducible over $\mathbb{Q}$ and whose roots are positive real numbers.

$\mathcal{F}_g^{\min}$: the subset of $\mathcal{F}_g$ of minimal trace polynomials.

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Lemma (B., Giangreco '22)

Let g be a positive integer. There exists a polynomial $\mathfrak{f}_g \in \mathcal{F}_g^{\min}$ and a real number n_g such that $\mathfrak{f}_g(t) > f(t)$ for any other polynomial $f \in \mathcal{F}_g$ and $t > n_g$.

In particular, $\mathfrak{f}_g$ is the maximal element of $\mathcal{F}_g^{\min}$ under the order relation:

$$f_1 \leq f_2 \iff f_2 - f_1 \text{ has non-negative leading coefficient.}$$

Main theorem: sketch of the proof

Theorem (B., Giangreco '22)

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3. $\#\mathcal{I}^0_{\max}(g, q)(\mathbb{F}_q) = h_g(1, \sqrt{q}) = \mathfrak{f}_g((\sqrt{q} + 1)^2)$;
4. maximality of $\mathfrak{f}_g \Rightarrow$ maximality of $\mathcal{I}^0_{\max}(g, q)$ within simple ordinary isogeny classes.

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When the set $\mathcal{F}_g^{\min}$ is known, we deduce $\mathfrak{f}_g$, hence N_g. Some examples[2]:

$\mathfrak{f}_g(t)$	$\mathfrak{f}_g(4)$	$\mathfrak{f}_g(0)$	Δ_g
$t-1$	3	-1	1
t^2-3t+1	5	1	5
t^3-5t^2+6t-1	7	-1	7^2
$t^4-7t^3+14t^2-8t+1$	1	1	$3^2\times 5^3$
$t^5-9t^4+28t^3-35t^2+15t-1$	11	-1	11^4
$t^6-11t^5+45t^4-84t^3+70t^2-21t+1$	13	1	13^5

[2] C. J. Smyth, Ann. Inst. Fourier Grenoble, 34, 1984

Further research

1. What can we say when q is an odd power of a prime?
   - $\rightarrow$ Is there a polynomial "parametrising" $\mathcal{I}^0_{\max}(g, q)$?
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2. maximal simple ordinary isogeny class $\overset{?}{\Leftrightarrow}$ maximal simple ~~ordinary~~ isogeny class
   - $\rightarrow$ The maximal simple isogeny class always has a irreducible Weil polynomial?

Thank you for your attention!

Questions?

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