Computing Riemann–Roch spaces for Algebraic Geometry codes

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joint with S. Abelard (Thales), A. Couvreur (Inria), G. Lecerf (LIX)

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Linear codes: from Reed–Solomon codes...

Linear code: $\mathbb{F}_q$–vector sub space of $\mathbb{F}_q^n$

$[n, k, d]_q$–code: code of length **n**, dimension **k** and minimum distance **d**

$$\left.\begin{array}{r}\text{dimension} \leftrightarrow \text{information} \\ \text{minimum distance} \leftrightarrow \text{correction capacity}\end{array}\right\} \quad k + d \leqslant n + 1$$

Singleton, 1964

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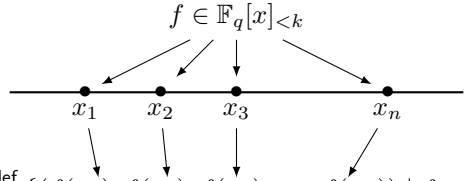
Reed–Solomon (RS) Codes Reed and Solomon, 1960



$$\mathrm{RS}_k(\mathbf{x}) \stackrel{\text{def}}{=} \{(f(x_1), f(x_2), f(x_3), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[x]_{<k}\}$$

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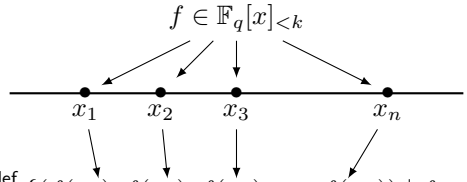

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- ✔ **Optimal parameters**
  $k + d = n + 1$.
- ✔ **Effective decoding algorithms**
  Berlekamp,1968.
- ⚠ **Drawback:** $n \leqslant q$.

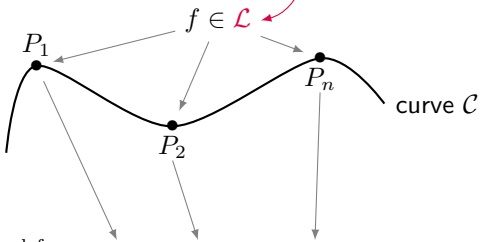
The bigger the q,
the less efficient the arithmetic.

...to Algebraic Geometry (AG) codes 🕮 Goppa, 1981

$\mathcal{P} = (P_1, P_2, \ldots, P_n)$

Vector space of functions on the curve

(Riemann–Roch space)



$$C_{\mathcal{C}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in L\}$$

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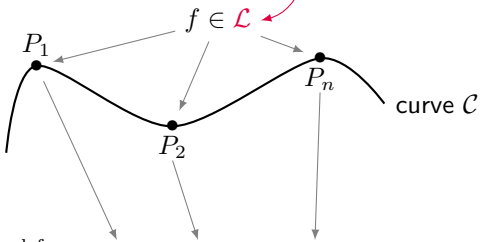

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Codes on a **curve** $\mathcal{C}$

- ✔ **Good parameters**
- ✔ **Efficient decoding algorithms**
- ✔ **Length** $> q$

$\#\mathcal{C}(\mathbb{F}_q) \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor$

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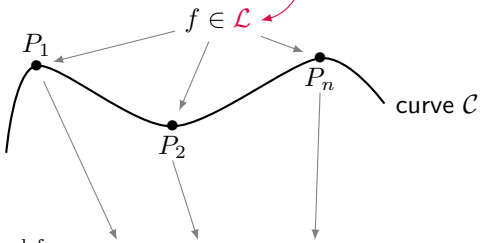

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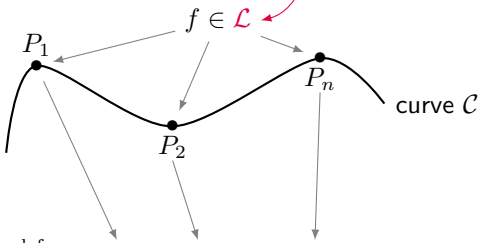

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AG codes are at distance g *from optimality*

AG codes: applications and construction

AG codes found application in (not exhaustive list)

- Constructing quantum error correcting codes[1]
  Necessary for the implementation of quantum computers.
- Secret sharing[2]
  Example: can have up to 500 players over $\mathbb{F}_{64}$ with AG codes from maximal curves, while need to work over a field with > 500 elements with RS codes.
- Verifiable computing[3]
  For zero–knowledge proofs, blockchain...

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TODAY: Computing Riemann–Roch spaces of curves.

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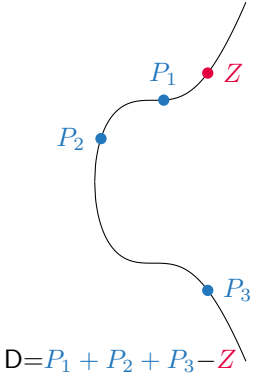
Riemann–Roch spaces of curves

A divisor on a curve $\mathcal{C}$: $D = \sum_{P \in \mathcal{C}} n_P P,\ n_P \in \mathbb{Z}$



The **Riemann–Roch space** $L(D)$ is the space of functions $\frac{G}{H} \in \mathbb{K}(\mathcal{C})$ such that:

- if $n_P < 0$ then P must be a zero of G (of multiplicity $\geqslant -n_P$)
- if $n_P > 0$ then P can be a zero of H (of multiplicity $\leqslant n_P$)
- G/H has no other poles outside the points P with $n_P > 0$

Here: Z must be a zero of G, the P_i can be zeros of H

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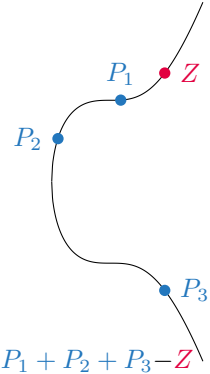

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Riemann–Roch Theorem $\rightsquigarrow$ dimension of $L(D) = \deg D + 1 - g$

where the degree of a divisor is $\deg D = \sum_P n_P \deg(P)$.

Toy example

Let $\mathcal{C} = \mathbb{P}^1$, $P = [0:1]$ and $Q = [1:1]$. Let $D = P - Q$, then

$$f \in L(D) \iff \begin{cases} \text{f has a zero of order at least } 1 \text{ at } Q, \\ \text{f can have a pole of order at most } 1 \text{ at } P, \\ \text{f has not other poles outside } P. \end{cases}$$

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$$g = 0, \deg D = 0 \xrightarrow[\text{Theorem}]{\text{Riemann–Roch}} \dim L(D) = \deg D + 1 - g = 1$$

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Strategy

Denominator H vanishes at P: $H(X,Y,1) \equiv 0 \mod X$
Numerator G vanishes at Q: $G(X,Y,1) \equiv 0 \mod X-1$
We retrieve the solution: $\frac{X-1}{X}$.

Brill–Noether method 1874

Notations:

- $(H) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(H) P$ – divisor of the zeros of H with multiplicity
- $D \geqslant D' \rightsquigarrow D - D' = \sum n_P P$ with $n_P \geqslant 0 \ \forall P$ ($D - D'$ is *effective*)

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Description of $L(D)$ for $\mathcal{C} : F(X, Y, Z) = 0$ a plane projective curve.

The non–zero elements are of the form $\frac{G_i}{H}$ where

- *H satisfies $(H) \geqslant D$*
- *H vanishes at any singular point of $\mathcal{C}$ with ad hoc multiplicity*
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The non–zero elements are of the form $\frac{G_i}{H}$ where

- H *satisfies* $(H) \geqslant D$
- H *satisfies* $(H) \geqslant \mathcal{A}$ *(we say that "H is adjoint to the curve")*
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Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degree δ, D a smooth divisor.

Step 1 : Compute the adjoint divisor $\mathcal{A}$

Step 2 : Compute the common denominator H

Step 3 : Compute $(H) - D$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

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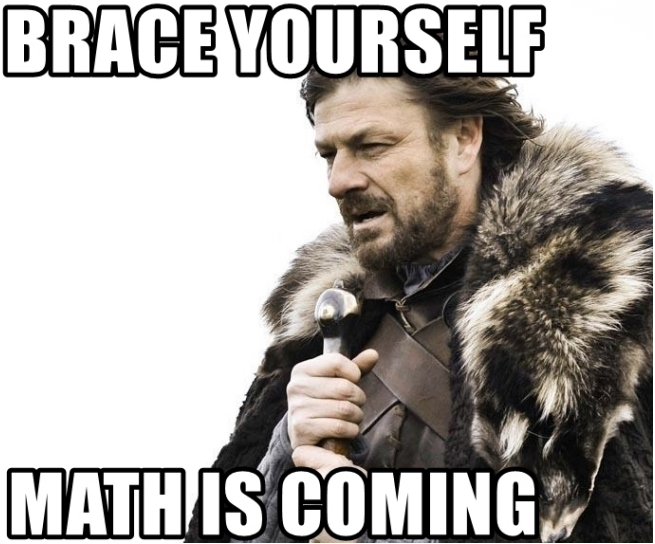
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MATH IS COMING

The adjoint divisor

Let $\mathcal{C} : F = 0$ be a plane curve. Let $P \in \mathrm{Sing}(\mathcal{C})$ of multiplicity m, w.l.o.g. $P = (0 : 0 : 1)$.

Definition

The local adjoint divisor is $\mathcal{A}_P = -\sum_{\mathcal{P}|P} \mathrm{val}_{\mathcal{P}} \left(\frac{dx}{F_y(x,y,1)} \right) \mathcal{P}$.

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$$F(x,y,1) = u(x,y) \prod_{i=1}^{m} (y - \varphi_i(x))$$

with $u \in \overline{\mathbb{K}}[[x,y]]$ invertible, φ_i Puiseux series of $F \in \overline{\mathbb{K}}[[x]][y]$.

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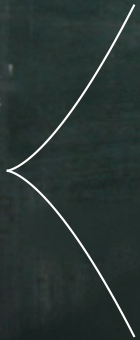

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The local adjoint divisor becomes

$$\mathcal{A}_P = -\sum_{\mathcal{P}|P} \mathrm{val}_t \left(\frac{e t^{e-1}}{F_y(X(t), Y(t), 1)} \right) \mathcal{P}.$$

Example

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$$\mathrm{val}_t \left(\frac{e t^{e-1}}{F_y(X(t), Y(t), 1)} \right) = \mathrm{val}_t \left(\frac{2t}{2t^3} \right) = \mathrm{val}_t \left(\frac{1}{t^2} \right) = -2$$

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<u>Adjoint condition:</u> $x = t^2 \Rightarrow dx = 2t,\ F_y = 2y$

$$\mathrm{val}_t \left(\frac{e t^{e-1}}{F_y(X(t), Y(t), 1)} \right) = \mathrm{val}_t \left(\frac{2t}{2t^3} \right) = \mathrm{val}_t \left(\frac{1}{t^2} \right) = -2$$

$(H) \geq \mathcal{A} \iff \mathrm{val}_t H(t^2, t^3) \geq 2$

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degree δ, D a smooth divisor .

Step 1 : Compute the adjoint divisor $\mathcal{A}$ ✔ $\leftarrow \tilde{O}(\delta^3)$

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Step 3 : Compute $(H) - D$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

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Let $d := \deg H$.

Condition $(H) \geqslant \mathcal{A} + D_+$

[4] $2 \leqslant \omega \leqslant 3$ is a feasible exponent for linear algebra ($\omega = 2.373$)

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$\rightsquigarrow$ linear system in the coefficients of H with $\deg \mathcal{A} + \deg D_+ \sim \delta^2 + \deg D_+$ equations,

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We showed that $d = \left\lceil \frac{(\delta-1)(\delta-2)+\deg D_+}{\delta} \right\rceil$ is enough

$\rightsquigarrow$ denominator H computed with $\tilde{O}((\delta^2 + \deg D_+)^\omega)$ operations.

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$\deg G_i = \deg H$, G_i prime with F and $(G_i) \geqslant (H) - D$

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Theorem (Abelard, B–, Couvreur, Lecerf – Journal of Complexity 2022)

The previous algorithm computes $L(D)$ with $\tilde{O}((\delta^2 + \deg D_+)^\omega)$ operations in $\mathbb{K}$.

What to take away?

0. Implementation of AG codes $\rightsquigarrow$ need to compute Riemann–Roch spaces $L(D)$
1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on G and H such that $G/H \in L(D)$
2. Puiseux series $\rightsquigarrow$ management of singular points of the curve
3. Linear Algebra $\rightsquigarrow$ Computing H and G in practice

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Main result

We can compute Riemann–Roch spaces of any plane curve with a good complexity exponent.

Future questions

- Computing Riemann–Roch spaces in positive "small" characteristic (in progress).

  Main obstacle: find an alternative tool to Puiseux series to handle the adjoint condition.

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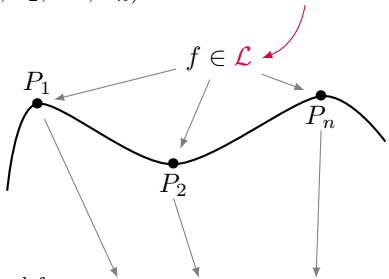
Riemann–Roch space of the curve



$$C_{\mathcal{C}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathcal{L}\}$$

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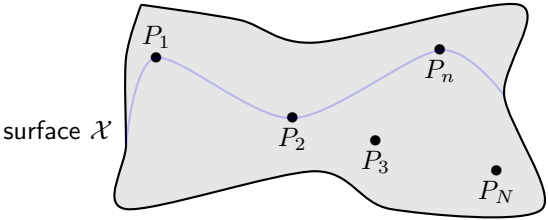

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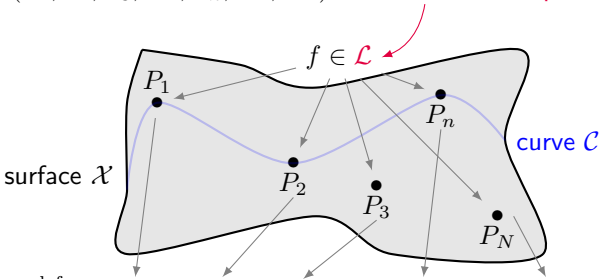
$\mathcal{P} = (P_1, P_2, P_3, \ldots, P_n, \ldots, P_N)$ Riemann–Roch space of the **surface**



$$C_{\mathcal{X}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), f(P_3), \ldots, f(P_n), \ldots, f(P_N)) \mid f \in \mathcal{L}\}$$

Codes on a **surface** $\mathcal{X}$

- ✔ **Length:** $N \sim q^2$
- ?? **Parameters & decoding**
  (✔ **very** particular cases)
- ✔ Local properties from curves on $\mathcal{X}$
  (*e.g.* local recoverability)

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Thank you for your attention!

Questions? e.berardini@tue.nl