Computing Riemann–Roch spaces for Algebraic Geometry codes

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joint with S. Abelard (Thales), A. Couvreur (Inria), G. Lecerf (LIX)

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Linear codes: from Reed–Solomon codes...

Linear code: $\mathbb{F}_q$–vector sub space of $\mathbb{F}_q^n$

$[n, k, d]_q$–code: code of length **n**, dimension **k** and minimum distance **d**

$$\left.\begin{array}{c}\text{dimension} \leftrightarrow \text{information} \\ \text{minimum distance} \leftrightarrow \text{correction capacity}\end{array}\right\} \quad k + d \leqslant n + 1$$ Singleton, 1964

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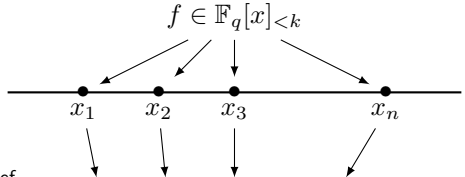
Reed–Solomon (RS) Codes Reed and Solomon, 1960



$$\mathrm{RS}_k(\mathbf{x}) \stackrel{\text{def}}{=} \{(f(x_1), f(x_2), f(x_3), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[x]_{<k}\}$$

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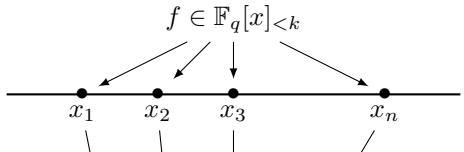

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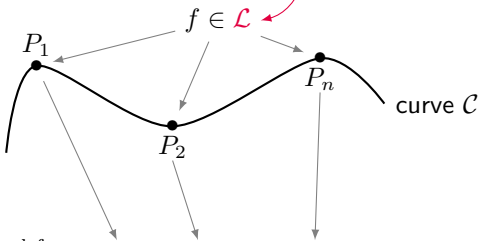
- ✔ **Optimal parameters**
  $k + d = n + 1$.
- ✔ **Effective decoding algorithms**
  Berlekamp,1968.
- ⚠ **Drawback:** $n \leqslant q$.

  The bigger the q,
  the less efficient the arithmetic.

...to Algebraic Geometry (AG) codes Goppa, 1981

$\mathcal{P} = (P_1, P_2, \ldots, P_n)$

Vector space of functions on the curve
(Riemann–Roch space)



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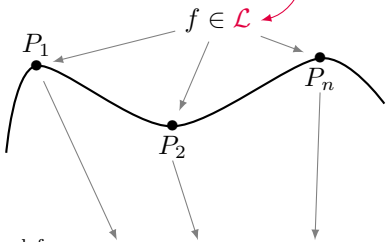

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Codes on a **curve** $\mathcal{C}$

- ✔ **Good parameters**
- ✔ **Efficient decoding algorithms**
- ✔ **Length** $> q$

$\#\mathcal{C}(\mathbb{F}_q) \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor$

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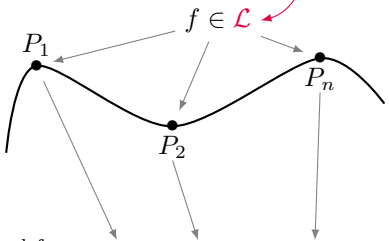

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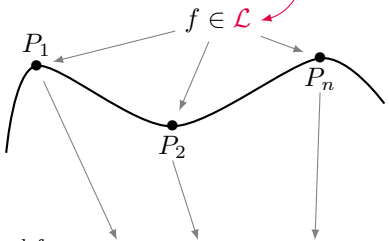

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AG codes are at distance g *from optimality*

AG codes: applications and construction

AG codes found application in (not exhaustive list)

- Constructing quantum error correcting codes[1]
- Secret sharing[2]
  Example: can have up to 500 players over $\mathbb{F}_{64}$ with AG codes from maximal curves, while need to work over a field with > 500 elements with RS codes
- Verifiable computing[3]

[1] La Guardia, Pereira, Quantum Information Processing, 2017
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Construction of **good AG codes** relies on $\begin{cases} \textbf{identify algebraic curves} \text{ suitable to the context,} \\ \textbf{design efficient algorithms} \text{ for implementation.} \end{cases}$

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$\rightsquigarrow$ computing Riemann–Roch spaces of curves

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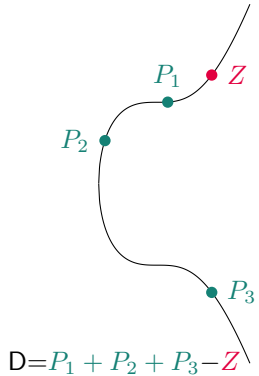
Riemann–Roch spaces of curves

A divisor on a curve $\mathcal{C}$: $D = \sum_{P \in \mathcal{C}} n_P P,\ n_P \in \mathbb{Z}$



The **Riemann–Roch space** $L(D)$ is the space of functions $\frac{G}{H} \in \mathbb{K}(\mathcal{C})$ such that:

- if $n_P < 0$ then P must be a zero of G (of multiplicity $\geqslant -n_P$)
- if $n_P > 0$ then P can be a zero of H (of multiplicity $\leqslant n_P$)
- G/H has no other poles outside the points P with $n_P > 0$

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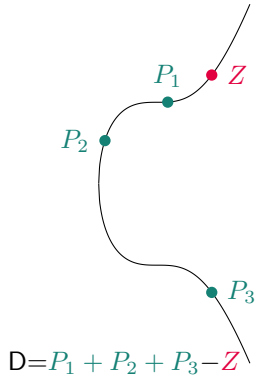

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Riemann–Roch Theorem $\rightsquigarrow$ dimension of $L(D) = \deg D + 1 - g$

where the degree of a divisor is $\deg D = \sum_P n_P \deg(P)$.

Toy example

Let $\mathcal{C} = \mathbb{P}^1$, $P = [0:1]$ and $Q = [1:1]$. Let $D = P - Q$, then

$$f \in L(D) \iff \begin{cases} \text{f has a zero of order at least } 1 \text{ at } Q, \\ \text{f can have a pole of order at most } 1 \text{ at } P, \\ \text{f has not other poles outside } P. \end{cases}$$

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$$g = 0, \deg D = 0 \xrightarrow[\text{Theorem}]{\text{Riemann–Roch}} \dim L(D) = \deg D + 1 - g = 1$$

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$$\rightarrow f \text{ generates the space of solutions.}$$

Strategy

Denominator H vanishes at P: $H(X, Y, 1) \equiv 0 \mod X$
Numerator G vanishes at Q: $G(X, Y, 1) \equiv 0 \mod X - 1$
We retrieve the solution: $\frac{X-1}{X}$.

Brill–Noether method

Notations:

- $(H) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(H) P$ – divisor of the zeros of H with multiplicity
- $D \geqslant D' \rightsquigarrow D - D' = \sum n_P P$ with $n_P \geqslant 0 \ \forall P$ ($D - D'$ is *effective*)
  We can always write $D = D_+ - D_-$ with D_+ and D_- two effective divisors.

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Description of $L(D)$ for $\mathcal{C} : F(X, Y, Z) = 0$ a plane projective curve.

The non–zero elements are of the form $\frac{G_i}{H}$ *where*

- H *satisfies* $(H) \geqslant D_+$
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The non–zero elements are of the form $\frac{G_i}{H}$ *where*

- H *satisfies* $(H) \geqslant D_+$
- H *satisfies* $(H) \geqslant \mathcal{A}$ *(we say that "H is adjoint to the curve")*
- $\deg G_i = \deg H$, G_i *prime with* F *and* $(G_i) \geqslant (H) - D$

How do we manage singular points?

✔ the adjoint divisor $\mathcal{A}$ "encodes" the singular points of $\mathcal{C}$ with their multiplicities

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ *a plane curve of degree* δ, D *a smooth divisor.*

Step 1 : Compute the adjoint divisor $\mathcal{A}$

Step 2 : Compute the common denominator H

Step 3 : Compute $(H) - D$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ *in terms of* H *and the* G_i.

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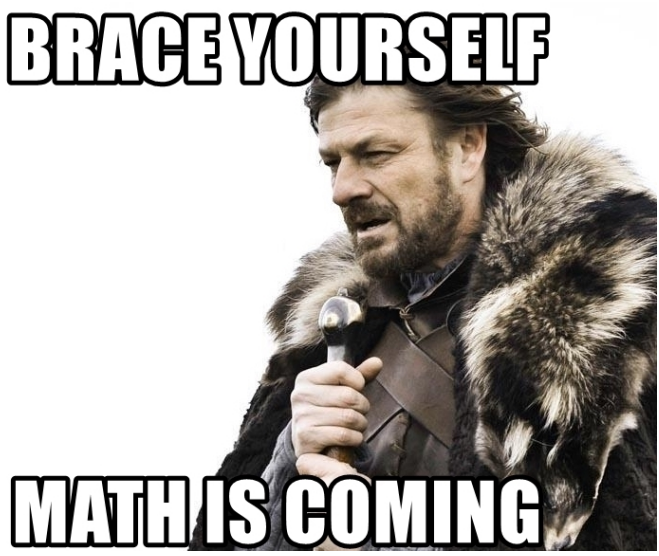
BRACE YOURSELF
MATH IS COMING

Warm up: adjoint divisor in the ordinary case

Definition

Let $\mathcal{C}$ be defined over a field $\mathbb{K}$, and let $P \in \mathrm{Sing}(\mathcal{C})$. The local adjoint divisor is

$$\mathcal{A}_P = -\sum_{\mathcal{P}|P} \mathrm{val}_{\mathcal{P}}\left(\frac{dx}{F_y(x,y,1)}\right)\mathcal{P}.$$

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Let $P \in \mathrm{Sing}(\mathcal{C})$ ordinary of multiplicity m, w.l.o.g. $P = (0:0:1)$. Then F locally factorises as

$$F(x,y,1) = u(x,y)\prod_{i=1}^{m}(y-\varphi_i(x))$$

with $u \in \overline{\mathbb{K}}[[x,y]]$ invertible, $\varphi_i(x) \in x\overline{\mathbb{K}}[[x]]$ and $\varphi_i'(0) \neq \varphi_j'(0)$.

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The *local adjoint divisor* becomes

$$\mathcal{A}_P = (m-1)\sum_{i=1}^{m}\mathcal{P}_i.$$

Adjoint condition via Puiseux series

Informally: Puiseux series are Laurent series that admit fractional exponents.

$F \in \mathbb{K}((x))[y]$ has $\deg F = d$ distinct roots in its field of Puiseux series and writes as

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A Rational Puiseux Expansion (RPE) is a pair $(X(t), Y(t)) = \left(\gamma t^e, \sum_{j=n}^{\infty} \beta_j t^j \right)$ *such that* $F(X(t), Y(t)) = 0$.

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Rational Puiseux Expansion of $F(x, y, 1)$ $\longleftrightarrow$ places of $\mathcal{C}$ in the chart $z = 1$

The adjoint divisor

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with $u \in \mathbb{K}[[x, y]]$ invertible and φ_i Puiseux series of $F \in \overline{\mathbb{K}}[[x]][y]$.

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In practice: algorithm for computing Puiseux series[4] $\rightsquigarrow$ $\mathcal{A}$ computed with $\tilde{O}(\delta^3)$ operations.

[4] A. Poteaux and M. Weimann, Annales Henri Lebesgue, 2021

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ *a plane curve of degree* δ, D *a smooth divisor .*

Step 1 : Compute the adjoint divisor $\mathcal{A}$ ✔ $\leftarrow \tilde{O}(\delta^3)$

Step 2 : Compute the common denominator H

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A basis of the Riemann–Roch space $L(D)$ *in terms of* H *and the* G_i.

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Find a denominator in practice: classical linear algebra

Let $d := \deg H$.

$$\text{Condition } (H) \geqslant \mathcal{A} + D_+$$

$\rightsquigarrow$ linear system with $\deg \mathcal{A} + \deg D_+ \sim \delta^2 + \deg D_+$ equations,

$\rightsquigarrow$ we retrieve H by Gauss elimination that costs

$$\tilde{O}((d\delta + \delta^2 + \deg D)^\omega) \text{ operations}^5 \text{ in } \mathbb{K}.$$

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We showed that $d = \left\lceil \frac{(\delta-1)(\delta-2)+\deg D_+}{\delta} \right\rceil$ is enough

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Second method: structured linear algebra

Condition $(H) \geqslant \mathcal{A}$

$$\rightsquigarrow \operatorname{val}_t(H(X(t), Y(t), 1) \geqslant -\operatorname{val}_t\left(\frac{et^{e-1}}{F_y(X(t), Y(t), 1)}\right)$$

(similar equations for the condition $(H) \geqslant D_+$)

The space of polynomials $H(x, y, 1)$ that satisfy these conditions is a $\mathbb{K}[x]$–module

$\rightsquigarrow$ computing a basis[6] costs $\tilde{O}((\delta^2 + \deg D)^\omega)$ operations in $\mathbb{K}$.

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Same complexity exponent but with some

Advantages:

- better complexity exponent over algebraically closed fields: $\tilde{O}((\delta^2 + \deg D)^{\frac{\omega+1}{2}})$,
- potential improvement in the future.

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Theorem (Abelard, B–, Couvreur, Lecerf – Journal of Complexity 2022)

The previous algorithm computes $L(D)$ *with* $\tilde{\mathcal{O}}((\delta^2 + \deg D_+)^\omega)$ *operations in* $\mathbb{K}$*.*

What to take away?

0. Implementation of AG codes $\rightsquigarrow$ need to compute Riemann–Roch spaces $L(D)$

1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on G and H such that $G/H \in L(D)$

2. Puiseux series $\rightsquigarrow$ management of *non–ordinary* singular points of the curve

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Main result

We can compute Riemann–Roch spaces of any plane curve with a good complexity exponent.

Future questions

- Computing Riemann–Roch spaces of non–ordinary curves in positive "small" characteristic (in progress).

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Thank you for your attention!

Questions? e.berardini@tue.nl