# Computing Riemann–Roch spaces for Algebraic Geometry codes

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joint with S. Abelard (Thales), A. Couvreur (Inria), G. Lecerf (LIX)

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## Linear codes: from Reed–Solomon codes...

Linear code: $\mathbb{F}_q$–vector sub space of $\mathbb{F}_q^n$

$[n, k, d]_q$–code: code of length **n**, dimension **k** and minimum distance **d**

$$\left.\begin{array}{r} \text{dimension} \leftrightarrow \text{information} \\ \text{minimum distance} \leftrightarrow \text{correction capacity} \end{array}\right\} \quad k + d \leqslant n + 1$$

Singleton, 1964

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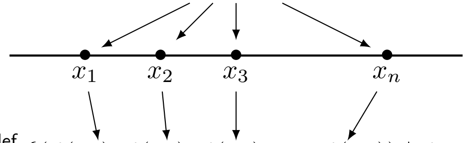
### Reed–Solomon (RS) Codes

Reed and Solomon, 1960

$$f \in \mathbb{F}_q[x]_{<k}$$




$$\mathrm{RS}_k(\mathbf{x}) \stackrel{\text{def}}{=} \{(f(x_1), f(x_2), f(x_3), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[x]_{<k}\}$$

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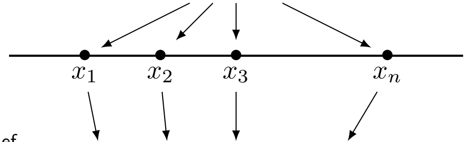


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- ✔ **Optimal parameters**
  $k + d = n + 1.$
- ✔ **Effective decoding algorithms**
  Berlekamp,1968.

⚠ **Drawback:** $n \leqslant q$.

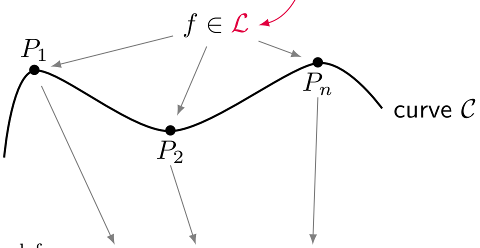
The more $q$ is big,
the less the arithmetic is efficient.

## ...to Algebraic Geometry (AG) codes Goppa, 1981

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

Vector space of functions on the curve

(Riemann–Roch space)




$$C_{\mathcal{C}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in L\}$$

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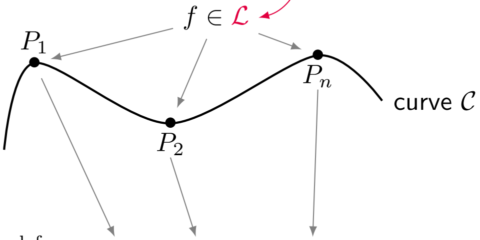


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Codes on a **curve** $\mathcal{C}$

- ✔ **Good parameters**
- ✔ **Efficient decoding algorithms**
- ✔ **Length** $> q$

$\#\mathcal{C}(\mathbb{F}_q) \leq q + 1 + g\lfloor 2\sqrt{q} \rfloor$

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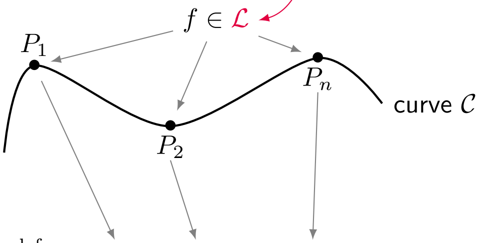


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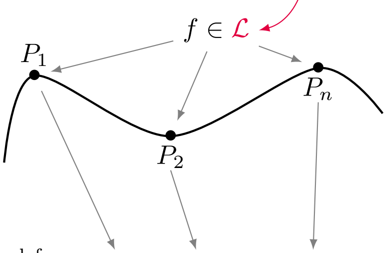


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Construction of **good AG codes** relies on $\begin{cases} \text{identify algebraic curves suitable to the context,} \\ \text{design efficient algorithms for implementation.} \end{cases}$

(**identify algebraic curves** suitable to the context, **design efficient algorithms** for implementation.)

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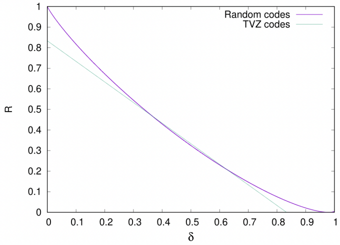

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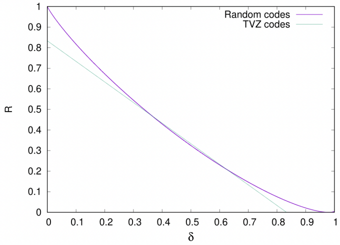


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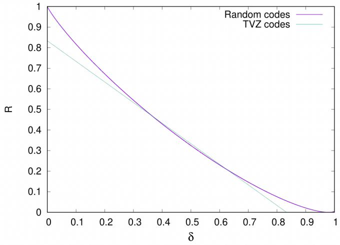


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XXIc: AG codes are used in new applications in information theory...

## Riemann–Roch spaces: AG codes and beyond

AG codes provide complexity gains in (not exhaustive list)

- Secret sharing[1]
  Example: can have up to 500 players over $\mathbb{F}_{64}$ with AG codes from maximal curves, while need to work over a field with $> 500$ elements with $\mathrm{RS}$ codes
- Verifiable computing[2]

$\rightsquigarrow$ computing large Riemann–Roch spaces of curves is necessary

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Can be used also for...

- Arithmetic operations on Jacobians of curves[3]
- Symbolic integration[4]

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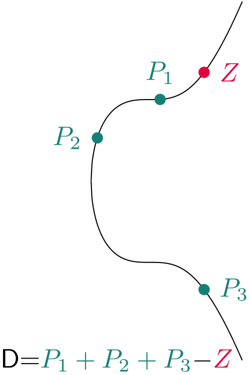
[3] K. Khuri-Makdisi, Mathematics of Computations, 2007
[4] J.H. Davenport, Intern. Symp. on Symbolic et Algebraic Manipulation, 1979

## Riemann–Roch spaces of curves

A divisor on a curve $\mathcal{C}$: $D = \sum_{P \in \mathcal{C}} n_P P,\ n_P \in \mathbb{Z}$




The **Riemann–Roch space** $L(D)$ is the space of functions $\frac{G}{H} \in \mathbb{K}(\mathcal{C})$ such that:

- if $n_P < 0$ then $P$ must be a zero of $G$ (of multiplicity $\geqslant -n_P$)
- if $n_P > 0$ then $P$ can be a zero of $H$ (of multiplicity $\leqslant n_P$)
- $G/H$ has no other poles outside the points $P$ with $n_P > 0$

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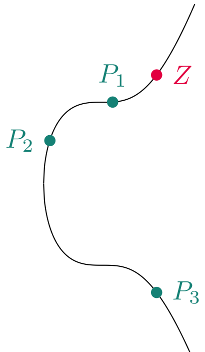


D=$P_1 + P_2 + P_3 - Z$

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**Riemann–Roch Theorem** $\rightsquigarrow$ dimension of $L(D) = \deg D + 1 - g$

where the degree of a divisor is $\deg D = \sum_P n_P \deg(P)$.

## Toy example

Let $\mathcal{C} = \mathbb{P}^1$, $P = [0:1]$ and $Q = [1:1]$. Let $D = P - Q$, then

$$f \in L(D) \iff \begin{cases} \text{f has a zero of order at least } 1 \text{ at } Q, \\ \text{f can have a pole of order at most } 1 \text{ at } P, \\ \text{f has not other poles outside } P. \end{cases}$$

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$\rightarrow f$ generates the space of solutions.

⚠ no explicit method to compute a basis of $L(D)$!
How do we solve the problem in general?

## Riemann–Roch problem: state of the art

**Geometric Method:**
(Brill–Noether theory∼1874)
- Goppa, Le Brigand–Risler (80's)
- Huang–Ierardi (90's)
- Khuri–Makdisi (2007)
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**Arithmetic Method:**
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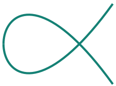
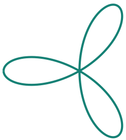
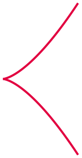
Ordinary/nodal curves: Las Vegas algorithm computing $L(D)$ in sub–quadratic time

Non–ordinary curves: ⚠ no explicit complexity exponent

## Brill–Noether method

Notations:

- $(H) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(H) P$ – divisor of the zeros of $H$ with multiplicity
- $D \geqslant D' \rightsquigarrow D - D' = \sum n_P P$ with $n_P \geqslant 0 \; \forall P$ ($D - D'$ is *effective*)
  We can always write $D = D_+ - D_-$ with $D_+$ and $D_-$ two effective divisors.

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**Description of $L(D)$ for $\mathcal{C} : F(X, Y, Z) = 0$ a plane projective curve.**

*The non–zero elements are of the form $\frac{G_i}{H}$ where*

- $H$ *satisfies* $(H) \geqslant D_+$
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series expansions of multi–set $((P_i)_i, n_i)$ $\quad\rightarrow\quad$ operations on divisors with negligible cost

## Sketch of the algorithm

**Input**

*$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degree $\delta$, $D$ a smooth divisor.*

**Step 1 :** Compute the adjoint divisor $\mathcal{A}$

**Step 2 :** Compute the common denominator $H$

**Step 3 :** Compute $(H) - D$

**Step 4 :** Compute the numerators $G_i$ (similar to Step 2)

**Output**

*A basis of the Riemann–Roch space $L(D)$ in terms of $H$ and the $G_i$.*

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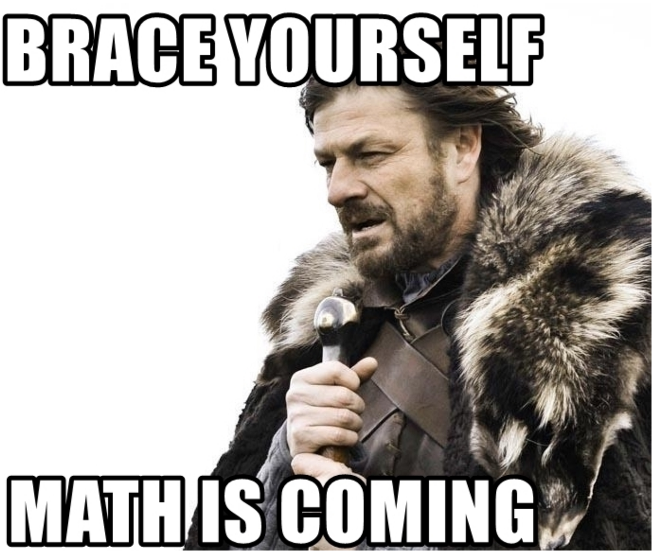
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MATH IS COMING

## Warm up: adjoint divisor in the ordinary case

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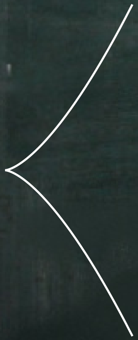
The *local adjoint divisor* becomes $\mathcal{A}_P = (m-1)\sum_{i=1}^{m}\mathcal{P}_i$.

## Example



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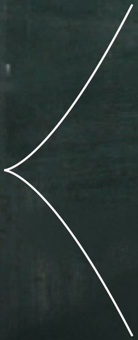

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We use Puiseux series!

## Adjoint condition via Puiseux series

Informally: Puiseux series are Laurent series that admit fractional exponents.

$F \in \mathbb{K}((x))[y]$ has $\deg F = d$ distinct roots in its field of Puiseux series and writes as

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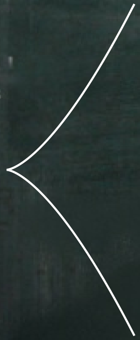
**In practice:** algorithm for computing Puiseux series[5] $\rightsquigarrow$ $\mathcal{A}$ computed with $\tilde{O}(\delta^3)$ operations.

[5] A. Poteaux and M. Weimann, Annales Herni Lebesgue, 2021

# Example



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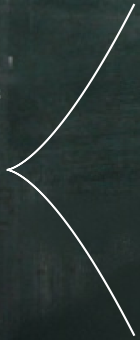

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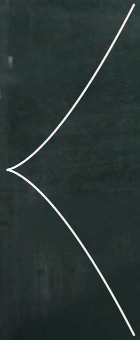

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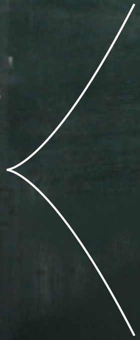

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$$(H) \geq \mathcal{A} \iff \operatorname{val}_t H(t^2, t^3) \geq 2$$

## Sketch of the algorithm

### Input

$\mathcal{C} : F(X, Y, Z) = 0$ *a plane curve of degree* $\delta$, $D$ *a smooth divisor .*

**Step 1 :** Compute the adjoint divisor $\mathcal{A}$ ✔ $\leftarrow \tilde{O}(\delta^3)$

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Let $d := \deg H$.

$$\text{Condition } (H) \geqslant \mathcal{A} + D_+$$

$\rightsquigarrow$ linear system with $\deg \mathcal{A} + \deg D_+ \sim \delta^2 + \deg D_+$ equations,

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$$\tilde{O}((d\delta + \delta^2 + \deg D)^\omega) \text{ operations}^6 \text{ in } \mathbb{K}.$$

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**How big is $d$?**

We showed that $d = \left\lceil \frac{(\delta-1)(\delta-2)+\deg D_+}{\delta} \right\rceil$ is enough

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## Second method: structured linear algebra

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The space of polynomials $H(x, y, 1)$ that satisfy these conditions is a $\mathbb{K}[x]$–module

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Same complexity exponent but with some

Advantages:

- better complexity exponent over algebraically closed fields: $\tilde{O}((\delta^2 + \deg D)^{\frac{\omega+1}{2}})$,
- potential improvement in the future.

[7] C.-P. Jeannerod, V. Neiger, É. Schost and G. Villard, J. Symbolic Comput. 2017

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### Theorem (Abelard, B–, Couvreur, Lecerf – Journal of Complexity 2022)

*The previous algorithm computes $L(D)$ with $\tilde{O}((\delta^2 + \deg D_+)^\omega)$ operations in $\mathbb{K}$.*

## What to take away?

0. Implementation of AG codes $\rightsquigarrow$ need to compute large Riemann–Roch spaces $L(D)$
1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on $G$ and $H$ such that $G/H \in L(D)$
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**Main result**

*We can compute Riemann–Roch spaces of any plane curve with a good complexity exponent.*

## Future questions

- Computing Riemann–Roch spaces of non–ordinary curves in positive “small” characteristic (in progress).

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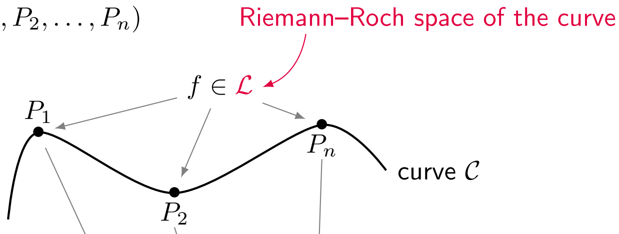
- Improving the complexity exponent in the non–ordinary case. (Sub–quadratic as in the ordinary case?)

  **Main obstacle:** linear algebra.

## AG codes: from curves to surfaces

$\mathcal{P} = (P_1, P_2, \ldots, P_n)$




$$C_{\mathcal{C}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathcal{L}\}$$

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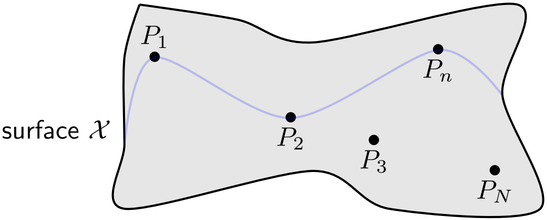

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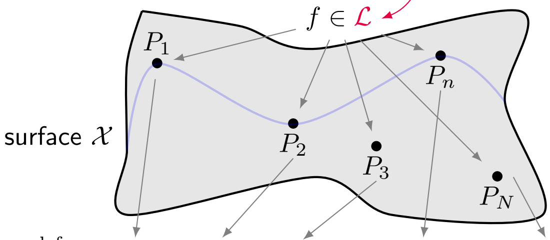
Riemann–Roch space of the **surface**

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- ✔ **Length:** $N \sim q^2$
- **??** **Parameters & decoding**
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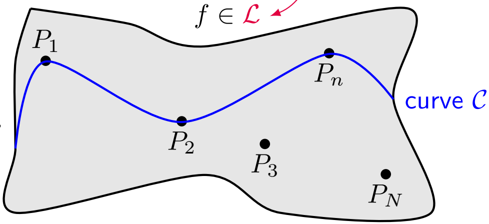


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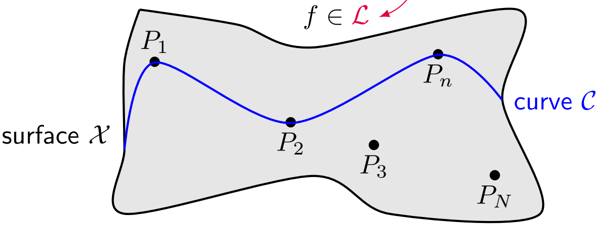


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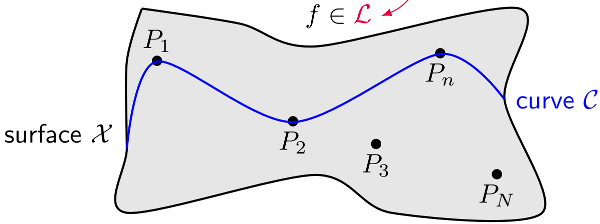
✔ Local properties from curves lying on $\mathcal{X}$
(*e.g.* local decoding, local recoverability)

## AG codes: from curves to surfaces

$\mathcal{P} = (P_1, P_2, P_3, \dots, P_n, \dots, P_N)$ Riemann–Roch space of the **surface**




Codes on a **surface** $\mathcal{X}$

- ✔ **Length:** $N \sim q^2$
- **??** **Parameters & decoding**
  (✔ **very** particular cases)

$$C_{\mathcal{X}}(\mathcal{L}, \mathcal{P}) \stackrel{\text{def}}{=} \{(f(P_1), f(P_2), f(P_3), \dots, f(P_n), \dots, f(P_N)) \mid f \in \mathcal{L}\}$$

Restriction to $\mathcal{C}$

$$C_{\mathcal{C}}(\mathcal{L}_{\mathcal{C}}, \mathcal{P}_{\mathcal{C}}) = \{(f(P_1), f(P_2), \dots, f(P_n)) \mid f \in \mathcal{L}\}$$

✔ Local properties from curves lying on $\mathcal{X}$
(*e.g.* local decoding, local recoverability)

- Can we develop a "Brill–Noether" theory for computing Riemann–Roch spaces of surfaces?

# Thank you for your attention!

Questions?

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