ON THE NUMBER OF RATIONAL POINTS ON CURVES LYING ON A SURFACE IN $\mathbb{P}^3$

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Outline of the presentation

1. Pre–existing results and motivations
2. Our strategy
3. Preliminaries: geometry of space curves
4. Technical details
5. Final result and open question

Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with q elements, and $\overline{\mathbb{F}}_q$ an algebraic closure of it.
The projective space $\mathbb{P}^n$ is the set of equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{0\}$ under the relation $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for every $\lambda \in \overline{\mathbb{F}}_q \setminus \{0\}$.

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Degree of a variety $\subset \mathbb{P}^3$ (examples):

$$S : (f = 0) \Rightarrow \deg S = \deg f \qquad \text{(Surfaces)}$$

$$\mathcal{C} : f = g = 0 \Rightarrow \deg \mathcal{C} = \deg f \times \deg g. \qquad \text{(Complete intersection)}$$

Existing bounds

Theorem [Hasse–Weil, 1948]

If C is an absolutely irreducible smooth curve of genus g defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$.

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Theorem [Homma, 2012]

If C is a non–degenerate curve defined over $\mathbb{F}_q$ of degree δ in $\mathbb{P}^n$, with $n \geq 3$, then $\#C(\mathbb{F}_q) \leq (\delta - 1)q + 1$.

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Theorem [Stöhr–Voloch, 1986]

Let $C/\mathbb{F}_q$ be an irreducible smooth curve of genus g and degree δ in $\mathbb{P}^n$. Let $\nu_1, \dots, \nu_{n-1}$ be its Frobenius orders (generically $\nu_i = i$). Then

$$\#C(\mathbb{F}_q) \leq \frac{1}{n}\left((\nu_1 + \cdots + \nu_{n-1})(2g - 2) + (q + n)\delta\right).$$

Stöhr and Voloch's strategy for plane curves

Let C be a plane curve of degree δ defined over $\mathbb{F}_q$. Denote by Φ the q–Frobenius morphism.

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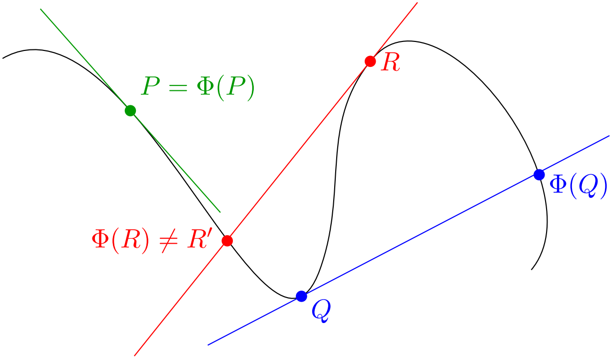

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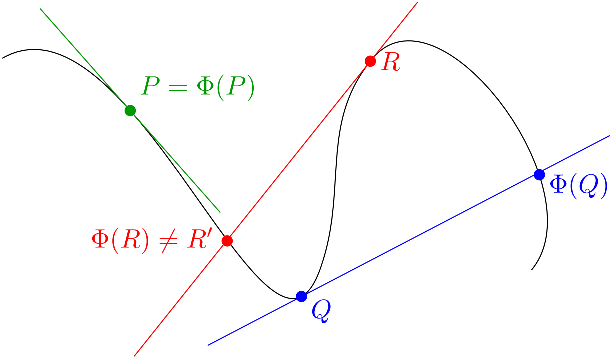

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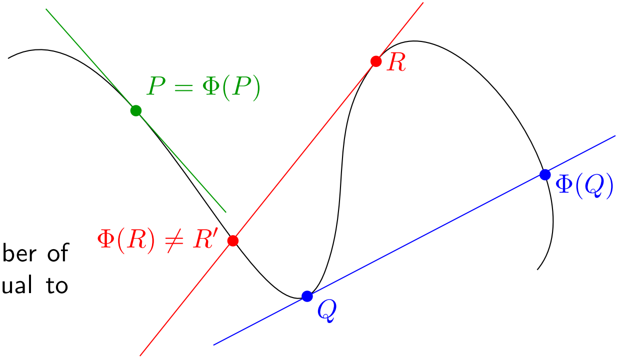

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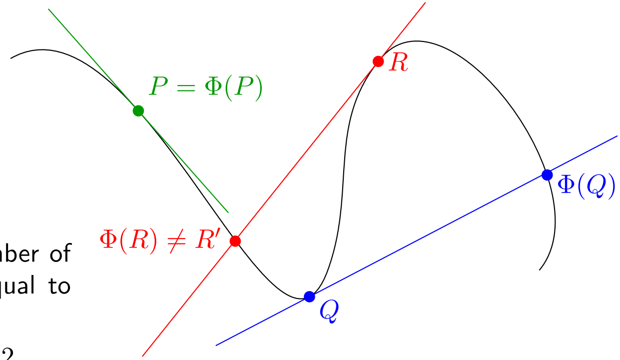

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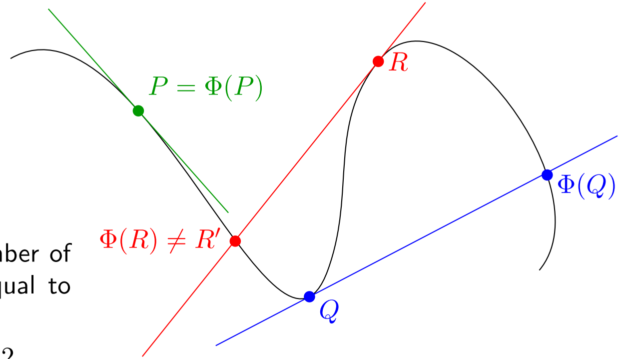

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If C has at least a non–flex point $(\Rightarrow \dim \mathcal{Z} = 0)$, then $\#C(\mathbb{F}_q) \leq \frac{1}{2}\delta(\delta + q - 1)$.

Ideas & Motivations

Let $C \subset S \hookrightarrow \mathbb{P}^n$ (via a very ample divisor).

Goal: bounding $\#C(\mathbb{F}_q)$ in terms of the embedding.

(features of the surface S and the ambient $\mathbb{P}^n$)

Main motivations:

- New bound for the number of rational points on projective curves.

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  Code from a surface S:

  $$\mathsf{C}(S, \mathcal{P}, \overset{\text{divisor}}{D}) = \{(f(P_1), \ldots, f(P_n)) \mid f \in \overset{\text{Riemann–Roch space}}{L(D)}\}$$

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Bounding the minimum distance of a code from a surface S	$\rightsquigarrow$	Bounding $\#C(\mathbb{F}_q)$ for the irreducible curves C on S
Better lower bound for the minimum distance	$\Longleftrightarrow$	Better upper bound for $\#C(\mathbb{F}_q)$

Strategy ($n = 3$)

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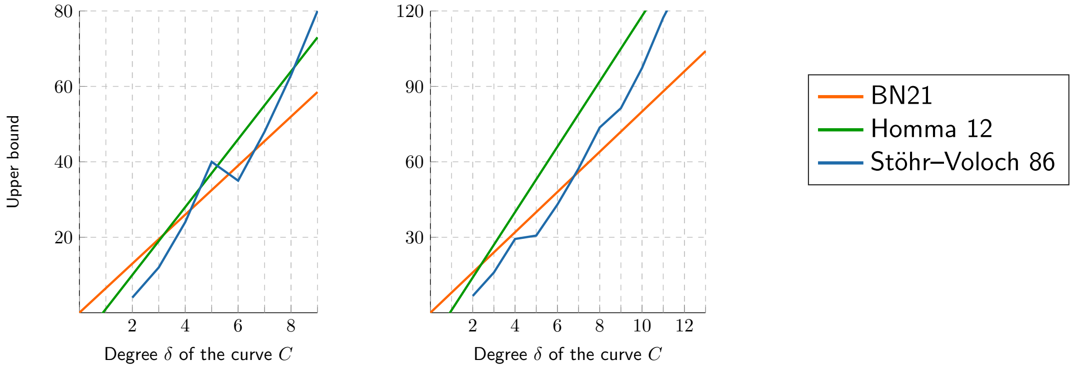
Comparisons with pre–existing bounds



(a) $q = 9$ and $d = 5$ (b) $q = 13$ and $d = 4$

Figure: Bounds on the number of $\mathbb{F}_q$–points on a non–plane curve C on a degree d surface $S \subset \mathbb{P}^3$.

$\rightarrow$ It is worth working on this bound!

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   ✔ $p \nmid d(d-1) \Rightarrow S$ is Frobenius classical.
2. C does not share any components with C_Φ.
   Counterexample: if S contains a $\mathbb{F}_q$–line L, then $L \subset C_\Phi$. The bound does not hold.

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If $C \cap C_\Phi$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi)}{\min_{P \in C(\mathbb{F}_q)} m_P(C \cap C_\Phi)} \leq \frac{\delta(d + q - 1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi) = 0$:

1. $\dim C_\Phi = 1$: in this case, the surface is said to be *Frobenius classical*;
   Counterexample: the Hermitian surface $X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} + T^{\sqrt{q}+1} = 0$ over $\mathbb{F}_q$.
   ✔ $p \nmid d(d-1) \Rightarrow S$ is Frobenius classical.
2. C does not share any components with C_Φ.
   Counterexample: if S contains a $\mathbb{F}_q$–line L, then $L \subset C_\Phi$. The bound does not hold.

Aim: understanding the components of the curve C_Φ for a Frobenius classical surface.

Osculating spaces and P-orders (Stöhr–Voloch theory 1)

Let $C \subset \mathbb{P}^3$ be an absolutely irreducible projective space curve defined over $\mathbb{F}_q$. Fix $P \in C$.
An integer j is a P–order if there exists a plane intersecting the curve C with multiplicity j at P.
If C is non–plane and P is non–singular, there are exactly four distinct P–orders:

$$j_0 = 0 < j_1 < j_2 < j_3.$$

Remark: $j_1 = 1 \Leftrightarrow C$ is non–singular at the point P.

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For almost every point $P \in C$, the sequence of P–orders is the same, say $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$.
There are only finitely many points such that $(j_0, j_1, j_2, j_3) \neq (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$, which are called the *Weierstrass points* of the curve.
Remark: $\varepsilon_1 = 1$ since almost every point is non–singular.

A curve is said to be classical if $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 2, 3)$ and non–classical otherwise.

Osculating spaces (Stöhr–Voloch theory 2)

Fix $P \in C \subset \mathbb{P}^3$ with P–orders $(0, j_1, j_2, j_3)$.

Osculating spaces: $T_P^{(i)}C = \bigcap\{\text{planes } H \text{ s.t. } m_P(C,H) \geq j_{i+1}\}$.

$$\begin{array}{ll} T_P^{(0)}C & = P, \\ \cap & \\ T_P^{(1)}C & = \text{tangent line for a non–singular point } P, \\ \cap & \\ T_P^{(2)}C & = \text{osculating plane of } C \text{ at } P. \\ \cap & \\ \mathbb{P}^3 & \end{array}$$

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Equation of the osculating plane $T_P^{(2)}C$:
$$\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$$

where $D_t^{(j)}$ are the *Hasse derivatives* with respect to a a local parameter t at P defined by

$$D_t^{(i)}t^k = \binom{k}{i} t^{k-i}.$$

Frobenius orders (Stöhr–Voloch theory 3)

Fix $P \in C \subset \mathbb{P}^3$ with P–orders $(0, j_1, j_2, j_3)$. Then $\Phi(P) \in T_P^{(2)}C$ if and only if

$$\begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$$

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Theorem [Stöhr–Voloch, 1986]

There exist integers $\nu_1 < \nu_2$ s.t. $\begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(\nu_1)}x_0 & D_t^{(\nu_1)}x_1 & D_t^{(\nu_1)}x_2 & D_t^{(\nu_1)}x_3 \\ D_t^{(\nu_2)}x_0 & D_t^{(\nu_2)}x_1 & D_t^{(\nu_2)}x_2 & D_t^{(\nu_2)}x_3 \end{vmatrix}$ is a nonzero function.
Choose them minimally with respect to the lexicographic order. Then $\{\nu_1, \nu_2\} \subset \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

The integers $\nu_0 = 0, \nu_1, \nu_2$ are called the *Frobenius orders* of C.

The curve C is Frobenius classical if $(\nu_1, \nu_2) = (1, 2)$, Frobenius non–classical otherwise.

Remark: No implication between Frobenius classical and classical.

Notations

Let $C \subset S$. Fix a generic point P on C, w.l.o.g. P is a non–singular point. We choose affine coordinates such that $P = (0,0,0)$ and S and C are locally given by

$$S : z = u(x,y), \qquad C : \begin{cases} y = g(x), \\ z = u(x, g(x)). \end{cases}$$

Denote by $(\tilde{x}, \tilde{y}, \tilde{z}) \overset{\text{def}}{=} \Phi(x,y,z)$. Note that $(\tilde{x}, \tilde{y}, \tilde{z}) = (x^q, y^q, z^q)$ if and only if $P \in C(\mathbb{F}_q)$.

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$$\Delta(i, j) \stackrel{\text{def}}{=} \det \begin{pmatrix} 1 & \tilde{x} & \tilde{y} & \tilde{z} \\ 1 & x & y & z \\ 0 & x^{(i)} & g^{(i)} & u(x, g(x))^{(i)} \\ 0 & 0 & g^{(j)} & u(x, g(x))^{(j)} \end{pmatrix}.$$

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Stöhr–Voloch Theorem $\Rightarrow \exists\, \nu_1, \nu_2$ s.t. $\Delta(\nu_1, \nu_2)$ is a nonzero function if C is non–plane.

Useful lemma

Aim: Understand the components of $C_\Phi = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

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Lemma [BN21]

Assume that we have $u^{(j)} = g^{(j)} u_y$ for every $j \geq \max\{2, \nu_1\}$. Then either $\nu_1 > 1$ and C is plane or $\nu_1 = 1$ and $\Phi(P) \notin T_P S$ for a generic point $P \in C$.

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Assume $\nu_1 > 1$. Since for $j \geq \nu_1$ we have $u^{(j)} = g^{(j)} u_y$, we obtain

$$\Delta(\nu_1, j) = \det \begin{pmatrix} 1 & \tilde{x} & \tilde{y} & \tilde{z} \\ 1 & x & y & z \\ 0 & 0 & g^{(\nu_1)} & g^{(\nu_1)} u_y \\ 0 & 0 & g^{(j)} & g^{(j)} u_y \end{pmatrix} = 0 \Rightarrow \Delta(\nu_1, j) = 0 \ \forall j \text{ (plane curve).}$$

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Assume $\nu_1 = 1$. Using that $u^{(j)} = g^{(j)} u_y$ for $j \geq 2$ we get

$$\Delta(1, j) = g^{(j)} \det \begin{pmatrix} 1 & \tilde{x} & \tilde{y} & \tilde{z} \\ 1 & x & y & z \\ 0 & 1 & g' & u_x + g' u_y \\ 0 & 0 & 1 & u_y \end{pmatrix} = g^{(j)} \underbrace{[(\tilde{x} - x) u_x + (\tilde{y} - y) u_y - (\tilde{z} - z)]}_{\neq 0 \text{ if } \Phi(P) \notin T_P S}.$$

Frobenius non–classical curves with $\nu_1 = 1$ are not components of C_Φ (1/2)

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Proposition [BN21]

Let C be a non–plane curve lying on a Frobenius classical surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then, for a generic point $P \in C$, we have $\Phi(P) \notin T_P S$.

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By contradiction, take P such that $\Phi(P) \in T_P S$. Since C is Frobenius non–classical we have

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$$\Rightarrow \ (x - \tilde{x})(g'u'' - g''g'u_y) - (y - \tilde{y})(u'' - g''u_y) = [(x - \tilde{x})g' - (y - \tilde{y})](u'' - g''u_y) = 0.$$

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Case 1: $g' = (y-\tilde{y})/(x-\tilde{x}) \Rightarrow \nu_1 > 1 \to$ contradiction. (C has $\nu_1 = 1$.)

Frobenius non–classical curves with $\nu_1 = 1$ are not components of C_Φ (2/2)

Case 2: $u'' - g''u_y = u_{yy}(g')^2 + 2g'u_{xy} + u_{xx} = 0.$

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$$g' = (y - \tilde{y})/(x - \tilde{x}) \quad \text{or} \quad g' = -u_{xy}/u_{yy}.$$

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(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ.

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(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ. However...

Proposition [BN21]

Assume that C is Frobenius non–classical with $\nu_1 > 1$ and $\delta \leq q$. Then C is plane.

Frobenius classical components of C_Φ

Recap: A component of C_Φ falls in one of the following cases:

- $\nu_1 > 1$: in this case, if it has $\delta \leq q$, it is plane;
- it is Frobenius classical, i.e. $\{\nu_1, \nu_2\} = \{1, 2\}$.

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Example of surface with highly reducible C_Φ

Over $\mathbb{F}_5$, consider the surface S defined by

$$f = \begin{array}{l} 2X_0X_1^2 + 2X_1^3 + 2X_0^2X_2 + 2X_0X_1X_2 + X_1^2X_2 + 2X_0X_2^2 + 3X_1X_2^2 \\ +3X_2^3 + 4X_0^2X_3 + X_0X_1X_3 + X_1^2X_3 + 2X_1X_2X_3 + 2X_2^2X_3 \\ +3X_0X_3^2 + 4X_1X_3^2 + X_2X_3^2. \end{array}$$

The curve C_Φ has degree 21 and is formed of 15 lines and one non–plane sextic $(\delta = q + 1)$.

Main result & Remarks

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

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- **Embedding entails arithmetic and geometric constraints on a variety:**
  For $\delta = 11$ and $d = 5$ over $\mathbb{F}_9$, C has genus at most 17 and $\#C(\mathbb{F}_q) \leq 72$.
  In `ManyPoints`, maximum curves of genus 16 and 17 have 74 $\mathbb{F}_9$–points.
  These record curves cannot lie on a Frob. classical surface, unless being a component of C_Φ.

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Our theorem essentially relies on the geometry of space curves and the intersection theory in $\mathbb{P}^3$.

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

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Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of Φ restricted to the curve C,
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$.

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
Remark: C_Φ was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the 1^{st} projection.

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Γ_C and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
$\rightarrow$ bound the number of rational points on C by a fraction of the intersection product $[\Gamma_C] \cdot [\mathcal{T}_S]$.

When $n \geq 4$, $[\Gamma_C] \cdot [\mathcal{T}_S] = 0$ while $\Gamma_C \cap \mathcal{T}_S \neq \varnothing$.
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Thank you for your attention!