Computing Riemann–Roch spaces for Algebraic Geometry codes

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I. Introduction to Algebraic Geometry codes (motivation)

II. Introduction to Riemann–Roch spaces

III. Computation of Riemann–Roch spaces

IV. Conclusion & future questions

Linear codes: from Reed–Solomon codes...

Linear code: $\mathbb{F}_q$–vector sub space of $\mathbb{F}_q^n$

$[n, k, d]_q$–code: code of length **n**, dimension **k** and minimum distance **d**

$$\left.\begin{array}{r} \text{dimension} \leftrightarrow \text{information} \\ \text{minimum distance} \leftrightarrow \text{correction capacity} \end{array}\right\} \quad k + d \leqslant n + 1$$

Singleton, 1964

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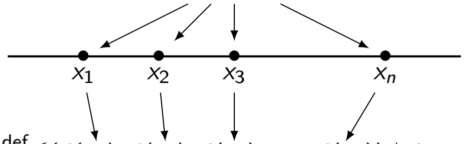
Reed–Solomon (RS) Codes

Reed and Solomon, 1960

$$f \in \mathbb{F}_q[x]_{<k}$$



$$\mathrm{RS}_k(\mathbf{x}) \overset{\text{def}}{=} \{(f(x_1), f(x_2), f(x_3), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[x]_{<k}\}$$

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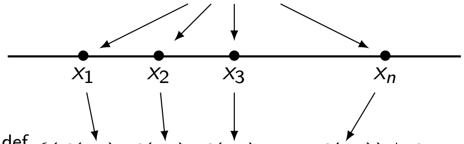

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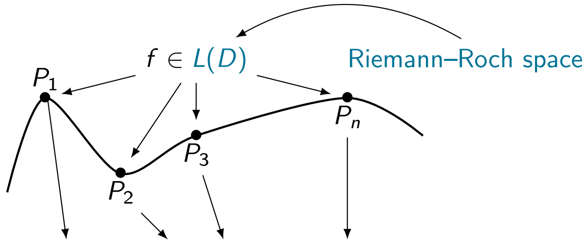
- ✔ Optimal parameters: $k + d = n + 1$.
- ✔ Effective decoding algorithms Berlekamp,1968
- ⚠ Drawback: $n \leqslant q$.

The more q is big,
the less the arithmetic is efficient.

...to Algebraic Geometry (AG) codes



$$\mathcal{C}((P_i)_i, D) := \{(f(P_1), f(P_2), f(P_3), \ldots, f(P_n)) \mid f \in L(D)\}$$

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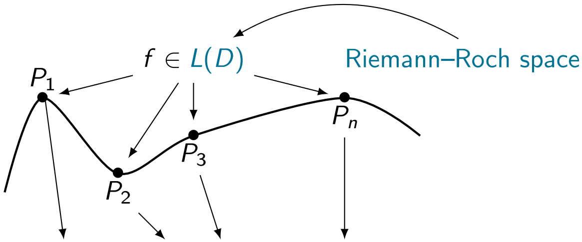

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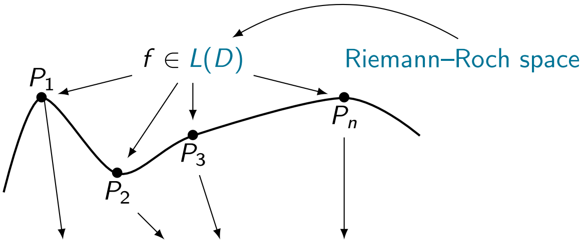

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Proposition

The parameters $[n, k, d]$ of AG codes satisfy

$$n + 1 - g \leq k + d \leq n + 1.$$

$\rightsquigarrow$ AG codes are a distance g from optimality

AG codes: long story short

1981: Goppa introduces AG codes from algebraic curves

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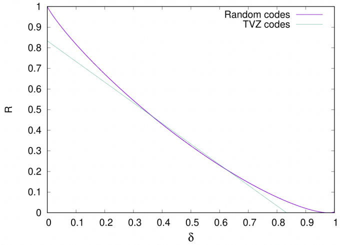

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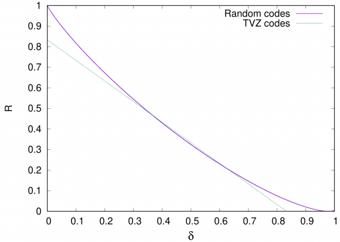

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$\hookrightarrow$ the most used curves are the ones for which Riemann–Roch spaces are already known (e.g. Hermitian curves)

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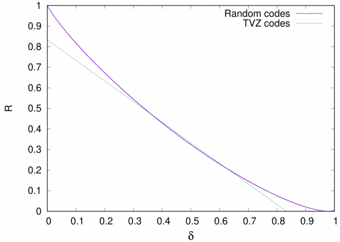

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XXIc: AG codes are used in new applications from information theory

Riemann–Roch spaces: AG codes and beyond

AG codes are involved in

- Secret sharing[1]
- Verifiable computing[2]
- ...

$\rightsquigarrow$ need of computing Riemann–Roch spaces of curves

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Can be used also for...

- Arithmetic operations on Jacobians of curves[3]
- Symbolic integration[4]

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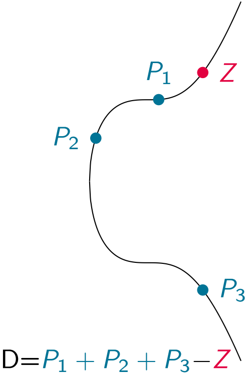
[3] K. Khuri-Makdisi, Mathematics of Computations, 2007
[4] J.H. Davenport, Intern. Symp. on Symbolic et Algebraic Manipulation, 1979

Riemann–Roch spaces of curves

A divisor on a curve $\mathcal{C}$: $D = \sum_{P \in \mathcal{C}} n_P P, \; n_P \in \mathbb{Z}$



The **Riemann–Roch space** $L(D)$ is the space of functions $\frac{G}{H} \in \mathbb{K}(\mathcal{C})$ such that:

- if $n_P < 0$ then P must be a zero of G (of multiplicity $\geqslant -n_P$)
- if $n_P > 0$ then P can be a zero of H (of multiplicity $\leqslant n_P$)
- G/H has no other poles outside the points P with $n_P > 0$

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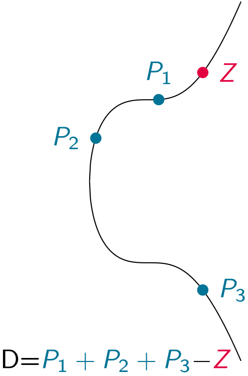

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Riemann–Roch Theorem $\rightsquigarrow$ dimension of $L(D) = \deg D + 1 - g$

where the degree of a divisor is $\deg D = \sum_P n_P \deg(P)$

Toy example

Let $\mathcal{C} = \mathbb{P}^1$, $P = [0:1]$ and $Q = [1:1]$. Let $D = P - Q$, then

$$f \in L(D) \iff \begin{cases} \text{f has a zero of order at least 1 at } Q \\ \text{f can have a pole of order at most 1 at } P \\ \text{f has not other poles outside } P \end{cases}$$

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$\rightarrow f$ generates the space of solutions

⚠ no explicit method to compute a basis of $L(D)$

How do we solve the problem in general?

Riemann–Roch problem: state of the art

Geometric Method:
(Brill–Noether theory∼1874)

- Goppa, Le Brigand–Risler (80's)
- Huang–Ierardi (90's)
- Khuri–Makdisi (2007)
- Le Gluher–Spaenlehauer (2018)
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Arithmetic Method:
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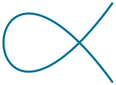
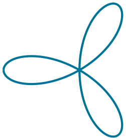
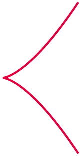
Ordinary/nodal curves: Las Vegas algorithm computing $L(D)$ in sub–quadratic time

Non–ordinary curves: ⚠ no explicit complexity exponent

Notations and hypotheses

$\mathcal{C} : F(x, y, z) = 0$ – plane curve, F absolutely irreducible of degree δ

$\mathrm{Sing}(\mathcal{C})$ – the singular points of $\mathcal{C}$, assumed in the affine chart $z = 1$

$(H) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(H) P$ – divisor of zeros of H with multiplicity

$D \geqslant D' \rightsquigarrow D - D' = \sum n_P P$ with $n_P \geqslant 0\ \forall P$ ($D - D'$ is *effective*)

We can always write $D = D_+ - D_-$ with D_+ and D_- two effective divisors

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⚠ well defined in characteristic 0 or positive "large"

Brill–Noether method

Description of $L(D)$ for $\mathcal{C} : F(X, Y, Z) = 0$ a plane projective curve.

The non–zero elements are of the form $\frac{G_i}{H}$ where

- H satisfies $(H) \geqslant D_+$
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- *H satisfies $(H) \geqslant \mathcal{A}$ (we say that "H is adjoint to the curve")*
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How do we represent divisors?

series expansions of multi–set representations $((P_i)_i, n_i)$ $\rightsquigarrow$ operations on divisors with negligible cost

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degree δ, D a smooth divisor.

Step 1 : Compute the adjoint divisor $\mathcal{A}$

Step 2 : Compute the common denominator H

Step 3 : Compute $(H) - D$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

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Warm up: adjoint divisor in the ordinary case

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Let $P \in \operatorname{Sing}(\mathcal{C})$ ordinary of multiplicity m, wlog $P = (0:0:1)$. Then F locally factorises as

$$F(x,y,1) = u(x,y)\prod_{i=1}^{m}(y - \varphi_i(x))$$

with $u \in \overline{\mathbb{K}}[[x,y]]$ invertible, $\varphi_i(x) \in x\overline{\mathbb{K}}[[x]]$ and $\varphi_i'(0) \neq \varphi_j'(0)$.

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The *local adjoint divisor* becomes $\quad \mathcal{A}_P = (m-1)\sum_{i=1}^{m}\mathcal{P}_i.$

Adjoint condition via Puiseux series

Let $F \in \mathbb{K}[x, y]$ be absolutely irreducible, monic in y and of degree d in y. $F \in \mathbb{K}((x))[y]$ has d distinct roots in $\overline{\mathbb{K}}\langle\langle x\rangle\rangle$, $\varphi_1, \ldots, \varphi_d$, and writes as

$$F = \prod_{i=1}^{d}(y - \varphi_i) = \prod_{i=1}^{d}\left(y - \sum_{j=n}^{\infty}\beta_{i,j}x^{j/e_i}\right).$$

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Definition

A Rational Puiseux Expansion (RPE) is a pair $(X(t), Y(t)) = \left(\gamma t^e, \sum_{j=n}^{\infty} \beta_j t^j\right)$ such that $F(X(t), Y(t)) = 0$.

Adjoint condition via Puiseux series

Let $F \in \mathbb{K}[x, y]$ be absolutely irreducible, monic in y and of degree d in y. $F \in \mathbb{K}((x))[y]$ has d distinct roots in $\overline{\mathbb{K}}\langle\langle x \rangle\rangle$, $\varphi_1, \ldots, \varphi_d$, and writes as

$$F = \prod_{i=1}^{d} (y - \varphi_i) = \prod_{i=1}^{d} \left(y - \sum_{j=n}^{\infty} \beta_{i,j} x^{j/e_i} \right).$$

We fix φ of degree e, ζ a primitive e-th root of unity. For $0 \leqslant k < e$ we can construct other e Puiseux series by replacing $x^{1/e}$ with $\zeta^k x^{1/e}$. They are all equivalent and represented by...

Definition

A Rational Puiseux Expansion (RPE) is a pair $(X(t), Y(t)) = \left(\gamma t^e, \sum_{j=n}^{\infty} \beta_j t^j\right)$ *such that* $F(X(t), Y(t)) = 0$.

Rational Puiseux Expansion of $F(x, y, 1)$ $\longleftrightarrow$ places of $\overline{\mathbb{K}}(\mathcal{C})$ in the chart $z = 1$

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$\mathcal{C} : y^2 - x^3 = 0$ in the chart $z = 1$

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⚠ the RPE are often defined over an extension of $\mathbb{K}$.
It is an algorithmic question to take the minimal extension of the field.

The adjoint divisor

Let $P \in \mathrm{Sing}(\mathcal{C})$ ~~ordinary~~, w.l.o.g. $P = (0:0:1)$. Then F locally factorises as

$$F(x, y, 1) = u(x, y) \prod_{i=1}^{m} (y - \varphi_i(x)),$$

with $u \in \mathbb{K}[[x, y]]$ invertible and φ_i Puiseux series of $F \in \overline{\mathbb{K}}[[x]][y]$.

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In practice: algorithm for computing Puiseux series[5] $\rightsquigarrow$ $\mathcal{A}$ computed with $\tilde{O}(\delta^3)$ operations.

[5] A. Poteaux and M. Weimann, Annales Herni Lebesgue, 2021

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$$\mathrm{val}_t\left(\frac{et^{e-1}}{F_y(X(t),Y(t),1)}\right) = \mathrm{val}_t\left(\frac{2t}{2t^3}\right) = \mathrm{val}_t\left(\frac{1}{t^2}\right) = -2$$

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H is adjoint $\iff \mathrm{val}_t H(t^2, t^3) \geq 2$

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degree δ, D a smooth divisor .

Step 1 : Compute the adjoint divisor $\mathcal{A}$ ✔ $\leftarrow \tilde{O}(\delta^3)$

Step 2 : Compute the common denominator H

Step 3 : Compute $(H) - D \leftarrow \tilde{O}((\delta^2 + \deg D_+)^2)$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

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Find a denominator in practice: classical linear algebra

Let $d := \deg H$.

$$\text{Condition } (H) \geqslant \mathcal{A} + D_+$$

$\rightsquigarrow$ linear system with $\deg \mathcal{A} + \deg D_+ \sim \delta^2 + \deg D_+$ equations

$\rightsquigarrow$ Gauss elimination costs

$$\tilde{O}((d\delta + \delta^2 + \deg D)^\omega) \text{ operations}^6 \text{ in } \mathbb{K}$$

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How big is d?

We showed that $d = \left\lceil \frac{(\delta-1)(\delta-2)+\deg D_+}{\delta} \right\rceil$ is enough

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Second method: structured linear algebra

Condition $(H) \geqslant \mathcal{A}$

$$\rightsquigarrow \mathrm{val}_t(H(X(t), Y(t), 1) \geqslant -\mathrm{val}_t\left(\frac{et^{e-1}}{F_y(X(t), Y(t), 1)}\right)$$

(similar equations for the condition $(H) \geqslant D_+$)

The space of polynomials $H(x, y, 1)$ that satisfy these conditions is a $\mathbb{K}[x]$–module

$\rightsquigarrow$ Computing a basis[7] costs $\tilde{O}((\delta^2 + \deg D)^\omega)$ operations

[7] C.-P. Jeannerod, V. Neiger, É. Schost and G. Villard, J. Symbolic Comput. 2017

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Same complexity exponent but with some

Advantages:

- better complexity exponent over algebraically closed fields: $\tilde{O}((\delta^2 + \deg D)^{\frac{\omega+1}{2}})$,
- potential improvement in the future.

[7] C.-P. Jeannerod, V. Neiger, É. Schost and G. Villard, J. Symbolic Comput. 2017

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Theorem (Abelard, B–, Couvreur, Lecerf – Journal of Complexity 2022)

The previous algorithm computes $L(D)$ with $\tilde{O}((\delta^2 + \deg D_+)^\omega)$ operations in $\mathbb{K}$.

What to take away?

0. Implementation of AG codes $\rightsquigarrow$ need of computing Riemann–Roch space $L(D)$
1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on G and H such that $G/H \in L(D)$
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Main result

We can compute Riemann–Roch spaces of any plane curve with a good complexity exponent.

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Thank you for your attention!

Questions? e.berardini@tue.nl