Computing Riemann–Roch spaces via Puiseux expansions

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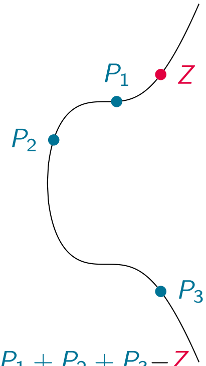
Projet funded by the Agence de l'Innovation de Défense

Journées Nationales de Calcul Formel
1 mars 2022

Riemann–Roch spaces of curves

A divisor on a curve $\mathcal{C}$: $D = \sum_{P \in \mathcal{C}} n_P P,\ n_P \in \mathbb{Z}$



D=$P_1 + P_2 + P_3 - Z$

The **Riemann–Roch space** $L(D)$ is the space of functions $\frac{G}{H} \in \mathbb{K}(\mathcal{C})$ such that:

- if $n_P < 0$ then P must be a zero of G (of multiplicity $\geqslant -n_P$)
- if $n_P > 0$ then P can be a zero of H (of multiplicity $\leqslant n_P$)
- G/H has no other poles outside the points P with $n_P > 0$

Here: Z must be a zero of G, the P_i can be zeros of H

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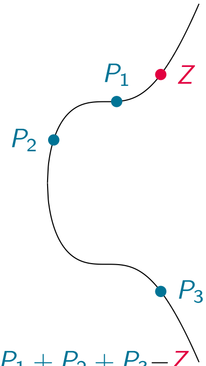

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Riemann–Roch Theorem $\rightsquigarrow$ dimension of $L(D) = \deg D + 1 - g$

where the degree of a divisor is $\deg D = \sum_P n_P \deg(P)$

Toy example

Let $\mathcal{C} = \mathbb{P}^1$, $P = [0:1]$ and $Q = [1:1]$. Let $D = P - Q$, then

$$f \in L(D) \iff \begin{cases} \text{f has a zero of order at least 1 at } Q \\ \text{f can have a pole of order at most 1 at } P \\ \text{f has not other poles outside } P \end{cases}$$

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$$g = 0, \deg D = 0 \xrightarrow[\text{Theorem}]{\text{Riemann–Roch}} \dim L(D) = \deg D + 1 - g = 1$$

$$\rightarrow f \text{ generates the space of solutions}$$

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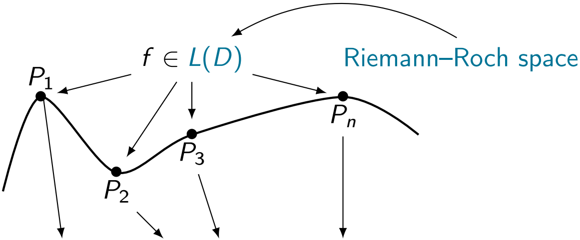
$\rightarrow f$ generates the space of solutions

⚠ no explicit method to compute a basis of $L(D)$

How do we solve the problem in general?

Riemann–Roch spaces: for what?

- Construction of algebraic geometry codes from curves



$$C((P_i)_i, D) := \{(f(P_1), f(P_2), f(P_3), \ldots, f(P_n)) \mid f \in L(D)\}$$

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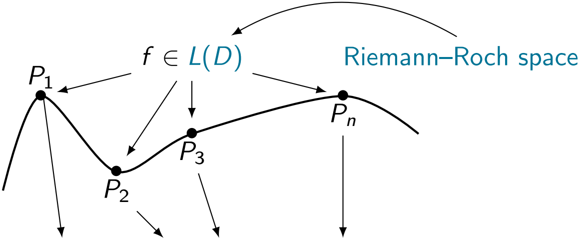

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- Arithmetic operations on Jacobians of curves[1]

[1] K. Khuri-Makdisi, Mathematics of Computations, 2007

Riemann–Roch problem: state of the art

Geometric Method:
(Brill–Noether theory∼1874)

- Goppa, Le Brigand–Risler (80's)
- Huang–Ierardi (90's)
- Khuri–Makdisi (2007)
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Arithmetic Method:
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Ordinary/nodal curves:	Las Vegas algorithm computing $L(D)$ in sub–quadratic time
Non–ordinary curves:	⚠ no explicit complexity exponent

Brill–Noether method

Notations:

- $(H) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(H) P$ – divisor of the zeros of H with multiplicity
- $D \geqslant D' \rightsquigarrow D - D' = \sum n_P P$ with $n_P \geqslant 0 \; \forall P$ ($D - D'$ is *effective*)

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Description of $L(D)$ for $\mathcal{C} : F(X, Y, Z) = 0$ a plane projective curve.

The non–zero elements are of the form $\frac{G_i}{H}$ where

- *H satisfies $(H) \geqslant D$*
- *H vanishes at any singular point of $\mathcal{C}$ with ad hoc multiplicity*
- $\deg G_i = \deg H$, *G_i prime with F and $(G_i) \geqslant (H) - D$*

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The non–zero elements are of the form $\frac{G_i}{H}$ where

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- *H satisfies $(H) \geqslant \mathcal{A}$ (we say that "H is adjoint to the curve")*
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How do we represent divisors?

series expansions of multi–set representations $((P_i)_i, n_i)$ $\rightsquigarrow$ operations on divisors with negligible cost

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ a plane curve of degre δ, D a smooth divisor.

Step 1 : Compute the adjoint divisor $\mathcal{A}$

Step 2 : Compute the common denominator H

Step 3 : Compute $(H) - D$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

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Warm up: adjoint divisor in the ordinary case

Definition

Let $P \in \mathrm{Sing}(\mathcal{C})$. The local adjoint divisor is

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$$F(x,y,1) = u(x,y) \prod_{i=1}^{m} (y - \varphi_i(x))$$

with $u \in \overline{\mathbb{K}}[[x,y]]$ invertible, $\varphi_i(x) \in x\overline{\mathbb{K}}[[x]]$ and $\varphi_i'(0) \neq \varphi_j'(0)$.

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The *local adjoint divisor* becomes

$$\mathcal{A}_P = (m-1)\sum_{i=1}^{m}\mathcal{P}_i.$$

Adjoint condition via Puiseux series

Let $F \in \mathbb{K}[x, y]$ be absolutely irreducible, monic in y and of degree d in y. $F \in \mathbb{K}((x))[y]$ has d distinct roots in $\overline{\mathbb{K}}\langle\langle x\rangle\rangle$, $\varphi_1, \ldots, \varphi_d$, and writes as

$$F = \prod_{i=1}^{d}(y - \varphi_i) = \prod_{i=1}^{d}\left(y - \sum_{j=n}^{\infty}\beta_{i,j}x^{j/e_i}\right).$$

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A Rational Puiseux Expansion (RPE) is a pair $(X(t), Y(t)) = \left(\gamma t^e, \sum_{j=n}^{\infty} \beta_j t^j\right)$ *such that* $F(X(t), Y(t)) = 0$.

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Rational Puiseux Expansion of $F(x, y, 1)$	$\longleftrightarrow$	places of $\overline{\mathbb{K}}(\mathcal{C})$ in the chart $z = 1$

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⚠ the RPE are often defined over an extension of $\mathbb{K}$.
It is an algorithmic question to take the minimal extension of the field.

The adjoint divisor

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In practice: algorithm for computing Puiseux series[2]

$\rightsquigarrow \mathcal{A}$ computed with $\tilde{O}(\delta^3)$ operations

[2] A. Poteaux et M. Weimann, Annales Herni Lebesgue, 2021

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ *a plane curve of degree* δ, D *a smooth divisor* .

Step 1 : Compute the adjoint divisor $\mathcal{A}$ ✓ $\leftarrow \tilde{O}(\delta^3)$

Step 2 : Compute the common denominator H

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Find a denominator in practice

Classical linear algebra

Let $d := \deg H$.

$$\text{Condition } (H) \geqslant \mathcal{A} + D$$

$\rightsquigarrow$ linear system with $\deg \mathcal{A} + \deg D \sim \delta^2 + \deg D$ equations

$\rightsquigarrow$ Gauss elimination costs

$$\tilde{O}((d\delta + \delta^2 + \deg D)^\omega) \text{ operations in } \mathbb{K}$$

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How big is d?

We showed that $d = \left\lceil \frac{(\delta-1)(\delta-2)+\deg D}{\delta} \right\rceil$ is enough

$\rightsquigarrow$ denominator computed with $\tilde{O}((\delta^2 + \deg D)^\omega)$ operations in $\mathbb{K}$

Second method: structured linear algebra

Condition $(H) \geqslant \mathcal{A}$

$$\rightsquigarrow \mathrm{val}_t(H(X(t), Y(t), 1) \geqslant \mathrm{val}_t\left(\frac{et^{e-1}}{F_y(X(t), Y(t), 1)}\right)$$

(similar equations for the condition $(H) \geqslant D$)

The space of polynomials $H(x, y, 1)$ that satisfy these conditions is a $\mathbb{K}[x]$–module

$\rightsquigarrow$ Computing a basis[3] costs $\tilde{O}((\delta^2 + \deg D)^\omega)$ operations

[3] C.-P. Jeannerod, V. Neiger, É. Schost et G. Villard, J. Symbolic Comput. 2017

Second method: structured linear algebra

Condition $(H) \geqslant \mathcal{A}$

$$\rightsquigarrow \mathrm{val}_t(H(X(t), Y(t), 1) \geqslant \mathrm{val}_t \left(\frac{et^{e-1}}{F_y(X(t), Y(t), 1)} \right)$$

(similar equations for the condition $(H) \geqslant D$)

The space of polynomials $H(x, y, 1)$ that satisfy these conditions is a $\mathbb{K}[x]$–module

$\rightsquigarrow$ Computing a basis[3] costs $\tilde{O}((\delta^2 + \deg D)^\omega)$ operations

Same complexity exponent but...

Advantages:

- better complexity exponent on algebraically closed fields
- potential improvement in the futur

[3] C.-P. Jeannerod, V. Neiger, É. Schost et G. Villard, J. Symbolic Comput. 2017

Sketch of the algorithm

Input

$\mathcal{C} : F(X, Y, Z) = 0$ *a plane curve of degree* δ, D *a smooth divisor .*

Step 1 : Compute the adjoint divisor $\mathcal{A}$ ✓ $\leftarrow \tilde{O}(\delta^3)$

Step 2 : Compute the common denominator H ✓ $\leftarrow \tilde{O}((\delta^2 + \deg D)^\omega)$

Step 3 : Compute $(H) - D$ ✓ $\leftarrow \tilde{O}(\delta^2 + \deg D)$

Step 4 : Compute the numerators G_i (similar to Step 2)

Output

A basis of the Riemann–Roch space $L(D)$ *in terms of* H *and the* G_i*.*

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A basis of the Riemann–Roch space $L(D)$ in terms of H and the G_i.

Theorem (Abelard, B., Couvreur, Lecerf – preprint 2021)

The previous algorithm computes $L(D)$ with $\tilde{\mathcal{O}}((\delta^2 + \deg D)^\omega)$ operations in $\mathbb{K}$.

What to take away?

1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on G and H such that $G/H \in L(D)$
2. Puiseux series $\rightsquigarrow$ management of *non–ordinary* singular points of the curve
3. Linear Algebra $\rightsquigarrow$ Computing H and G in practice

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1. Brill–Noether method $\rightsquigarrow$ necessary and sufficient conditions on G and H such that $G/H \in L(D)$
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3. Linear Algebra $\rightsquigarrow$ Computing H and G in practice

Main result

Las Vegas algorithm computing $L(D)$ with $\tilde{\mathcal{O}}((\delta^2 + \deg D)^\omega)$ operations.

Future questions

- Computing Riemann–Roch spaces of non–ordinary curves in positive “small” characteristic
- Implementing the algorithm
- Improving the complexity exponent in the non–ordinary case (sub–quadratic?)

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Merci de votre attention !

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