# Bound on the minimum distance of algebraic geometry codes over surfaces

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joint work with Yves Aubry, Fabien Herbaut, Marc Perret

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## Evaluation codes

Let $V/\mathbb{F}_q$ be an algebraic variety and set $V(\mathbb{F}_q) = \{P_1, \ldots, P_n\}$.
Let $D$ be a divisor on $V$ such that $\mathrm{Supp}(D) \cap V(\mathbb{F}_q) = \emptyset$.
Consider the Riemann-Roch space

$$L(D) = \{f \in \mathbb{F}_q^*(V) \mid (f) + D \geq 0\} \cup \{0\}.$$

## Definition:

*The code $\mathcal{C}(V, D)$ is defined to be the image of the evaluation map*

$$\begin{aligned} \mathrm{ev} : \quad L(D) &\longrightarrow \mathbb{F}_q^n \\ f &\longmapsto (f(P_1), \ldots, f(P_n)). \end{aligned}$$

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- Abelian surfaces (Aubry, B, Herbaut, Perret, 2019)

## Algebraic surfaces: basic notions and notations

$X$ smooth, projective, absolutely irreducible algebraic surface defined over the finite field $\mathbb{F}_q$

$D \in \mathrm{Div}(X)$, a formal sum of irreducible curves on $X$

$D \in \mathrm{Div}(X)$ is *nef* if $D.C \geq 0$ for every irreducible curve $C$ on $X$

Linear equivalence: $D \sim D' \iff D - D' = (f)$

$\mathrm{Pic}(X) = \mathrm{Div}(X)/\sim$

$\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, its rank is called the Picard number

$\cdot : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ the intersection pairing

- if $C$ and $D$ meet transversally then $C.D = \#(C \cap D)$
- symmetric and additive
- it depends only on the linear equivalence classes

# Evaluation codes from algebraic surfaces

Let $X$ be a surface and $G$ a rational divisor on $X$.
Consider the Riemann-Roch space

$$L(G) = \{f \in \mathbb{F}_q^*(X) \mid (f) + G \geq 0\} \cup \{0\}.$$

## Definition:

*Set $X(\mathbb{F}_q) = \{P_1, \ldots, P_n\}$. The code $\mathcal{C}(X, G)$ is defined to be the image of the evaluation map*

$$\begin{aligned} \mathrm{ev}: \quad L(G) &\longrightarrow \mathbb{F}_q^n \\ f &\longmapsto (f(P_1), \ldots, f(P_n)). \end{aligned}$$

# Length, Dimension, Minimum Distance

$$n = ?$$

$$\dim(\mathcal{C}(X, G)) = ?$$

$$d \geq ?$$

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For $f \in L(G) \setminus \{0\}$, $N(f) :=$ number of zero coordinates of $\mathrm{ev}(f)$

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dimension of the <u>Riemann-Roch space</u> $\Rightarrow$ <u>$\dim(\mathcal{C}(X, G))$</u>

upper bound for <u>$N(f)$</u> $\Rightarrow$ lower bound for <u>the minimum distance</u>

## A lower bound for the price of two upper bounds

Let $f \in L(G) \setminus \{0\} = \{f \in \mathbb{F}_q^*(X) \mid Z(f) - P(f) + G \geq 0\}$.
We consider the effective divisor

$$D_f = G + Z(f) - P(f) = \sum_{i=1}^{k} n_i D_i$$

where every $D_i$ is an irreducible curve of arithmetic genus $\pi_i$ and $n_i > 0$.
For $f \in L(G) \setminus \{0\}$ we have

$$\boxed{\#Z(f) = N(f) \leq \sum_{i=1}^{k} \#D_i(\mathbb{F}_q)}$$

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$$\left.\begin{array}{c}\text{upper bound for } k \\ + \\ \text{upper bound for } \#D_i(\mathbb{F}_q)\end{array}\right\} \Rightarrow \text{lower bound for } d$$

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- Riemann-Roch Theorem:
  $\dim_{\mathbb{F}_q} L(G) + \dim_{\mathbb{F}_q} L(K_X - G) = \frac{1}{2} G.(G - K_X) + 1 + p_a + s(G)$

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$$N(f) := \text{number of zero coordinates of } \mathrm{ev}(f)$$

$$d = \#X(\mathbb{F}_q) - \max_{f \in L(G)\setminus\{0\}} N(f)$$

$$N(f) \leq \sum_{i=1}^{k} \#D_i(\mathbb{F}_q)$$

## On the number of rational points of algebraic curves

Set $m = \lfloor 2\sqrt{q} \rfloor$.

Theorem: (Serre-Weil, 1983)

*Let $C$ be an absolutely irreducible smooth curve of genus $g$ defined over the finite field $\mathbb{F}_q$. Then*

$$\#C(\mathbb{F}_q) \leq q + 1 + gm.$$

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Theorem: (Aubry-Perret, 2004)
*Let $f : D \to C$ be a surjective flat morphism between the irreducible curve $D$ of arithmetic genus $\pi_D$ and the smooth absolutely irreducible curve $C$ of genus $g_C$ defined over $\mathbb{F}_q$. Let $\bar{r}$ be the number of absolutely irreducible components of $D$. Then*

$$|\#D(\mathbb{F}_q) - \#C(\mathbb{F}_q)| \leq (\bar{r} - 1)q + m(\pi_D - g_C)$$

## Bound for $N(f)$

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**Lemma:**

*Let $A$ be an ample divisor on $X$. Then we have*

1. *If* $\mathrm{NS}(X) = < A >$ *and* $G = rA$ *then we have*
   - (i) $\sum_{i=1}^{k} \bar{r}_i \leq r$,
   - (ii) *(Voloch-Zarzar)* $k \leq r$.
2. *Otherwise,*
   - (i) $\sum_{i=1}^{k} \bar{r}_i \leq G.A$,
   - (ii) $k \leq G.A$.

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2. *Otherwise,*
   - *if* $K_X$ *is nef then* $\sum_{i=1}^{k} \pi_i \leq \alpha(G, A) + k,$
   - *if* $K_X$ *is anti-ample then* $\sum_{i=1}^{k} \pi_i \leq (G.A)^2/2A^2 + k/2,$

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+ $\sum_{i=1}^{k} \pi_i - k \leq$...

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$$N(f) \leq q \sum_{i=1}^{k} \bar{r}_i + k + m \sum_{i=1}^{k} \pi_i$$

$$d = \#X(\mathbb{F}_q) - \max_{f \in L(G)\setminus\{0\}} N(f).$$

## Bound for the minimum distance

Theorem: (Aubry, B., Herbaut, Perret)

*The minimum distance of the codes $\mathcal{C}(X,G)$ satisfies:*

1. *If* $\mathrm{NS}(X) = < A >$ *and* $G = rA$ *then*

$$d \geq \#X(\mathbb{F}_q) - r(q+1) - m - mr^2A^2/2 - mrA.K_X/2.$$

2. *Otherwise,*

$$d \geq \begin{cases} \#X(\mathbb{F}_q) - G.A(q+1+m) - m\alpha(G,A) \text{ if } K_X \text{ is nef,} \\ \#X(\mathbb{F}_q) - G.A(q+1+m/2) - m(G.A)^2/2A^2 \text{ if } -K_X \text{ is ample,} \end{cases}$$

*where* $\alpha(G,A) = (G.A)^2/2A^2 + G.K_X/2.$

## Length, Dimension, Minimum Distance

$$n = \#X(\mathbb{F}_q)$$

$$\dim(\mathcal{C}(X,G)) \geq \frac{1}{2} G.(G - K_X) + 1 + p_a$$

$$d \geq \begin{cases} \#X(\mathbb{F}_q) - r(q+1) - m - mr(rA^2 + A.K_X)/2, & \text{if } \mathrm{NS}(X) = < A >, \\ \#X(\mathbb{F}_q) - G.A(q+1+m) - m\alpha(G,A), & \text{if } K_X \text{ is nef}, \\ \#X(\mathbb{F}_q) - G.A(q+1+m/2) - m(G.A)^2/2A^2, & \text{if } K_X \text{ is anti-ample.} \end{cases}$$

# Cubic surfaces in $\mathbb{P}^3$: the case of Voloch and Zarzar

Let $X$ be a cubic surface in $\mathbb{P}^3$.
Its arithmetic genus is $p_a(X) = \binom{d-1}{3} = \binom{2}{3} = 0$.
Let $L$ be an hyperplane in $X$: $L$ is ample and $L^2 = d = 3$.
The canonical divisor is $K_X = (d-4)L = -L$ and it is anti ample.

Take $G = rL$ and $A = L$. In this setting we have:

| | $K_X$ | G | A |
|---|---|---|---|
| $K_X$ | 3 | -3r | -3 |
| G | | $3r^2$ | 3r |
| A | | | 3 |

For the code $\mathcal{C}(X, rL)$ we get: $\dim \mathcal{C}(X, rL) \geq 3r(r+1)/2 + 1$,

$$d \geq \begin{cases} \#X(\mathbb{F}_q) + m(3r/2 - 1) - r(q+1) - 3mr^2/2, & \text{if } \mathrm{NS}(X) =< L >, \\ \#X(\mathbb{F}_q) - 3r(q+1+m/2) - 3mr^2/2, & \text{otherwise.} \end{cases}$$

## Fibrations

**Definition:**

*Let $f : S \to B$ be a $\pi_0$ fibration:*

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- $\pi_0 = 0 \to$ ruled surfaces;
- $\pi_0 = 1 \to$ elliptic surfaces;
- $\pi_0 \geq 2 \to$ surfaces of general type.

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For the code $\mathcal{C}(S, G)$ we get ($K_S$ is nef):

$$\boxed{\begin{gathered}\dim(\mathcal{C}(S, G)) \geq \frac{1}{2} G.(G - K_X) + 1 + p_a, \\ d \geq \#S(\mathbb{F}_q) - G.A(q + 1 + m) - m\alpha(G, A)\end{gathered}}$$

where $\alpha(G, A) = (G.A)^2/2A^2 + G.K_X/2$.

## The geometry of fibrations

Every divisor on $S$ can be uniquely written as a sum of horizontal and fiber curves.

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$$N(f) \leq h(\#B(\mathbb{F}_q) - mg_B - q) + m\sum_{i=1}^{k} \pi_i + q\sum_{i=1}^{k} \bar{r}_i + v$$

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$$N(f) \leq h(\#B(\mathbb{F}_q) - mg_B - q - 1) + m\alpha(G, A) + mk + qG.A + k.$$

# Codes wars: a new bound

Theorem: (Aubry, B., Herbaut, Perret)
*The minimum distance of the code $\mathcal{C}(S, G)$ satisfies*

$$d \geq \#S(\mathbb{F}_q) + (q + 1 + mg_B - \#B(\mathbb{F}_q)) - G.A(q + 1 + m) - m\alpha(G, A).$$

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Compare with

$$d \geq \#S(\mathbb{F}_q) - G.A(q + 1 + m) - m\alpha(G, A)$$

the new bound is better than the first and improves as $g_B$ grows and $\#B(\mathbb{F}_q)$ stays low.

## Abelian surfaces strike back

Let $X$ be an abelian surface defined over $\mathbb{F}_q$.

Its arithmetic genus is $p_a(X) = -1$ and $K_X = 0$.

Let $A$ be an ample divisor on $X$ and let $G = rA$.

For the code $\mathcal{C}(X, rA)$ we get:

$$\boxed{\begin{gathered}\dim \mathcal{C}(X, rA) \geq r^2 A^2/2, \\ d \geq \#X(\mathbb{F}_q) - rA^2(q + 1 + m) - mr^2 A^2/2.\end{gathered}}$$

# Number of rational points of curves over abelian surfaces

Let $X$ be an abelian surface.

Rational Points: set $m := \lfloor 2\sqrt{q} \rfloor$

- for $D$ an irreducible curve on $X$ of arithmetic genus $\pi$ we have

$$\#D(\mathbb{F}_q) \leq q + 1 - \mathrm{Tr}(X) + |\pi - 2|m$$

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Write $k = k_1 + k_2$ where

$$k_1 = \#\{D_i \mid \pi_i > \ell\}$$

$$k_2 = \#\{D_i \mid \pi_i \leq \ell\}$$

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$$\begin{aligned} k_1 = \#\{D_i \mid \pi_i > \ell\} \to &\quad \#D(\mathbb{F}_q) \leq q + 1 - \mathrm{Tr}(X) + (\pi - 2)m \\ k_2 = \#\{D_i \mid \pi_i \leq \ell\} \to &\quad \#D(\mathbb{F}_q) \leq \pi - 1 \end{aligned}$$

$$N(f) \leq k_1(q + 1 - \mathrm{Tr}(X) - 2m) + m \sum_{i=1}^{k_1} \pi_i + k_2(\ell - 1)$$

## Bound for $N(f)$

Lemma:

1. $k_2 \le r\sqrt{\frac{A^2}{2}} - k_1\sqrt{\ell}$,
2. $k_1\sqrt{\ell} \le r\sqrt{\frac{A^2}{2}}$,
3. $\sum_{i=1}^{k_1} \pi_i \le \left(r\sqrt{A^2/2} - k_1\sqrt{\ell}\right)^2 + r\sqrt{2A^2\ell} + (1-\ell)k_1$.

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$$N(f) \le \phi(k_1),$$

$$\phi(k_1) := m\ell k_1^2 + k_1\left(q + 1 - \mathrm{Tr}(X) - m(\ell+1) - mr\sqrt{2A^2\ell} - \sqrt{\ell}(\ell-1)\right)$$

$$+ mA^2r^2/2 + mr\sqrt{2A^2\ell} + r\sqrt{A^2/2}(\ell-1),$$

$$k_1 \in \left[1, \sqrt{\frac{A^2}{2\ell}}r\right].$$

# Codes wars: the last bound

We have:

$$N(f) \leq \begin{cases} \phi\left(\sqrt{\frac{A^2}{2\ell}}r\right) \text{ if } \sqrt{\frac{2\ell}{A^2}} \leq r \leq \frac{\sqrt{2}(q+1-\mathrm{Tr}(X)-m-\sqrt{\ell}(\ell-1))}{m\sqrt{A^2\ell}}, \\ \phi(1) \text{ otherwise.} \end{cases}$$

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Theorem: (Aubry, B., Herbaut, Perret)
*Let $X$ be a simple abelian surface of trace $\mathrm{Tr}(X)$ such that every irreducible curve on it has arithmetic genus $\pi > \ell$, for a positive integer $\ell$. Then the minimum distance $d$ of the code $\mathcal{C}(X, rA)$ satisfies:*

$$\boxed{d \geq \#X(\mathbb{F}_q) - r\sqrt{\frac{A^2}{2\ell}}\,(q+1-\mathrm{Tr}(X)+(\ell-1)m)}$$

*if* $\sqrt{\frac{2\ell}{A^2}} \leq r \leq \frac{\sqrt{2}(q+1-\mathrm{Tr}(X)-m-\sqrt{\ell}(\ell-1))}{m\sqrt{A^2\ell}}$, *otherwise*

$$\boxed{d \geq \#X(\mathbb{F}_q) - (q+1-\mathrm{Tr}(X)) - m(r^2A^2/2-1) - r\sqrt{\frac{A^2}{2}}(\ell-1).}$$

## Comparing the bounds

Compare

$$d \geq \#X(\mathbb{F}_q) - r\sqrt{\frac{A^2}{2\ell}}\,(q + 1 - \mathrm{Tr}(X) + (\ell - 1)m)$$

with

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$$d \geq \#X(\mathbb{F}_q) - r\sqrt{\frac{A^2}{2\ell}}\,(q + 1 - \mathrm{Tr}(X) + (\ell - 1)m)$$

$$d_{min} - \#X(\mathbb{F}_q) \underset{q \to \infty}{\sim} -r\sqrt{\frac{A^2}{2\ell}}q.$$

Remark: the bound for $\ell = 2$ is better than the one for $\ell = 1$!

Question: There exist abelian surfaces which do not contain absolutely irreducible curves of arithmetic genus 0, 1 nor 2?

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YES!

# Abelian surfaces without curves of low genus: starting point

Lemma:

*An abelian surface $X$ contains no absolutely irreducible curves of arithmetic genus* $0$, $1$ *nor* $2$ $\iff$ *$X$ is simple and not isogenous to a Jacobian surface.*

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Theorem: (Weil)

*Let $(X, \lambda)$ be a principally polarized abelian surface defined over the finite field $k$. Then $(X, \lambda)$ is either*

1. *the polarized Jacobian of a genus 2 curve over $k$,*
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Abelian surfaces that might have the property we are searching for:

- Weil restrictions of polarized elliptic curves over a quadratic extension of $k$,
- abelian surfaces defined over $k$ that do not admit a principal polarization.

## Abelian surfaces containing no curves of genus 0, 1 nor 2

Proposition: (Aubry, B., Herbaut, Perret)

(i) *Let $X$ be an abelian surface defined over $\mathbb{F}_q$ which does not admit a principal polarization. Then $X$ does not contain absolutely irreducible curves of arithmetic genus* 0, 1 *nor* 2.

(ii) *Let $q = p^e$. Let $E$ be and elliptic curve defined over $\mathbb{F}_{q^2}$ of trace $\mathrm{Tr}(E/\mathbb{F}_{q^2})$. Let $X$ be the $\mathbb{F}_{q^2}/\mathbb{F}_q$-Weil restriction of the elliptic curve $E$. Then $X$ does not contain absolutely irreducible curves defined over $\mathbb{F}_q$ of arithmetic genus* 0, 1 *nor* 2 *if and only if one of the following cases holds:*

(1) $\mathrm{Tr}(E/\mathbb{F}_{q^2}) = 2q - 1$;

(2) $p > 2$ *and* $\mathrm{Tr}(E/\mathbb{F}_{q^2}) = 2q - 2$;

(3) $p \equiv 11 \mod 12$ *or* $p = 3$, $q = \square$ *and* $\mathrm{Tr}(E/\mathbb{F}_{q^2}) = q$;

(4) $p = 2$, $q \neq \square$ *and* $\mathrm{Tr}(E/\mathbb{F}_{q^2}) = q$;

(5) $q = 2$ *or* $q = 3$, *and* $\mathrm{Tr}(E/\mathbb{F}_{q^2}) = 2q$.

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# A frame has no name

- Our bounds seem to confirm the idea that surfaces with small Picard number are suitable for obtaining good codes;
- Codes from surfaces with canonical divisor anti-ample seem interesting as well;
- Fibrations on curves of high genus and with few rational points could give good codes as well (this case gives the best bound on the minimum distance).

## Some ideas for the "sequel"...

I) Fibrations. Provide examples of algebraic smooth curves of high genus and with few rational points, and of fibrations on these curves.

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I) Fibrations. Provide examples of algebraic smooth curves of high genus and with few rational points, and of fibrations on these curves.

II) Abelian surfaces. Under which condition(s) an abelian surface does not contain absolutely irreducible genus 3 curves as well?

III) Lunch: Sounds like a good idea too.

# Thank you for your attention!

(Questions?)

*He who asks a question is a fool for five minutes;*
*he who does not ask a question remains a fool forever.*

Confucius